Stability of undercompressive shock profiles

Peter HOWARD Kevin ZUMBRUN

February 8, 2008

Abstract

Using a simplified pointwise iteration scheme, we establish nonlinear phase-asymptotic orbital stability of large-amplitude Lax, undercompressive, overcompressive, and mixed under–overcompressive type shock profiles of strictly parabolic systems of conservation laws with respect to initial perturbations $|u_0(x)| \le E_0(1+|x|)^{-3/2}$ in $C^{0+\alpha}$, E_0 sufficiently small, under the necessary conditions of spectral and hyperbolic stability together with transversality of the connecting profile. This completes the program initiated by Zumbrun and Howard in [53], extending to the general undercompressive case results obtained for Lax and overcompressive shock profiles in [51], [30], [53], [55], [44], [37]–[41], and for special undercompressive profiles in [31]–[32], [23]. In particular, together with spectral results of [59], our results yield nonlinear stability of large-amplitude undercompressive phase-transitional profiles near equilibrium of Slemrod's model [50] for van der Waal gas dynamics or elasticity with viscosity–capillarity.

1 Introduction

In the series of papers [55], [37]–[41], Zumbrun and Mascia–Zumbrun, building on methods introduced in [53, 23], have established nonlinear $L^1 \cap H^3 \to L^p$ (resp. $L^1 \cap H^2 \to L^p$) orbital stability, $p > 1$, of large-amplitude Lax-type shock profiles of systems of conservation laws with viscosity (resp. relaxation), by a simple shock-tracking argument using mainly $L^q \to L^p$ bounds on the linearized solution operator. For precursors of this method, see, e.g., [18], [26]–[27], [31]–[32], [53], [23]. See also the alternative arguments carried out in [51], [30] for small-amplitude Lax-type shock profiles of systems with artificial (Laplacian) viscosity, and in [53], [44] for large-amplitude Lax- or overcompressive-type profiles of systems with general, possibly degenerate viscosity.

The purpose of the present work is to point out that a simple pointwise version of the argument of [55], [37]–[41] may be applied also to under-, over-, and mixed under–overcompressive shock profiles of strictly parabolic systems, giving a simple and unified treatment of shock stability independent of the amplitude or type of the connecting profile, depending only on the necessary, Evans-function condition established in [53], [14], [4], [56]–[57], equivalent to spectral and hyperbolic stability plus transversality of the connecting profile as a solution of the associated traveling-wave ordinary differential equation (ODE). In particular, we obtain for the first time nonlinear stability of general undercompressive profiles such as arise in phase-transitional gas dynamics and elasticity [46]–[50] or multiphase flow [1]–[3], [25], extending results obtained for special undercompressive profiles in [31]–[32], [23].

Moreover, the slight additional detail afforded by our pointwise description is sufficient to give also convergence of the phase shift, or phase-asymptotic orbital stability (definition recalled below), which was lacking in [55], [37]–[41]. This completes the one-dimensional stability program initiated in [53], at least for systems with strictly parabolic viscosity, giving a complete characterization of stability analogous to that obtained by Sattinger in the scalar case [45]. The assumption of strict parabolicity is physically appropriate for applications to phase-transitional shock waves, at least in one dimension; see Remark 4. Indeed, together with the spectral analysis of [59], our results yield one-dimensional nonlinear stability of large-amplitude phase-transitional profiles near equilibrium of Slemrod's model for van der Waals gas dynamics or elasticity with viscosity–capillarity, one of only two large-amplitude stability results that have so far been obtained for physical models (the other being stability of profiles of isentropic gamma-law gas dynamics with $\gamma = 1$ [36]). Stability of undercompressive profiles for systems with degenerate viscosity remains an interesting open problem.

Consider a traveling-wave solution

$$u(x,t) = \bar{u}(x - st), \qquad \lim_{z\to\pm\infty} \bar{u}(z) = u_{\pm}, \tag{1}$$

or "shock profile", of a system of conservation laws

$$u_t + f(u)_x = (B(u)u_x)_x, \tag{2}$$

x, $t \in \mathbb{R}$, u, $f \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times n}$, corresponding to an "ideal", or discontinuous shock wave

$$u(x,t) = \begin{cases} u_- & x \le st, \\ u_+ & x > st, \end{cases} \tag{3}$$

of the associated hyperbolic system

$$u_t + f(u)_x = 0. \tag{4}$$

Without loss of generality (changing to coordinates moving with the shock), take $s = 0$, so that (1) becomes a stationary, or standing-wave solution convenient for stability analysis.

Following [53], [57], we make the standard assumptions:

(H0) $f, B \in C^2$.

(H1) $Re\, \sigma(B) > 0$.

(H2) $\sigma(f'(u_\pm))$ real, distinct, and nonzero.

(H3) $Re\, \sigma(-ikf'(u_\pm) - k^2 B(u_\pm)) < -\theta k^2$ for all real k, some $\theta > 0$.

(H4) There exists a solution $\bar{u}$ of (1)–(2), nearby which the set of all solutions connecting the same values $u_\pm$ forms a smooth manifold $\{\bar{u}^\delta\}$, $\delta \in \mathcal{U} \subset \mathbb{R}^\ell$, $\bar{u}^0 = \bar{u}$.

Definition 1. *An ideal shock* (3) *is classified as undercompressive, Lax, or overcompressive type according as $i - n$ is less than, equal to, or greater than* 1*, where i, denoting the sum of the dimensions i_- and i_+ of the center–unstable subspace of $df(u_-)$ and the center–stable subspace of $df(u_+)$, represents the total number of characteristics incoming to the shock.*

A viscous profile (1) *is classified as pure undercompressive type if the associated ideal shock is undercompressive and $\ell = 1$, pure Lax type if the corresponding ideal shock is Lax type and $\ell = i - n$, and pure overcompressive type if if the corresponding ideal shock is overcompressive and $\ell = i - n$, ℓ as in (H4) and i as in Definition 1. Otherwise it is classified as mixed under–overcompressive type; see [32], [53].*

Pure Lax type profiles are the most common type arising in standard gas dynamics, while pure over- and undercompressive type profiles arise in magnetohydrodynamics (MHD) and phase-transitional models. Mixed under–overcompressive profiles are also possible, as described in [32], [53], but seldom encountered; indeed, we do not know a physical example. In the pure Lax or undercompressive case, $\{\bar{u}^\delta\} = \{\bar{u}(\cdot - \delta)\}$ is just the set of all translates of the base profile $\bar{u}$, whereas in other cases it involves also deformations of $\bar{u}$. For further discussion of existence, structure, and classification of viscous profiles, see, e.g., [32], [53], [38]–[40], [56]–[58], and references therein.

Definition 2. *The profile $\bar{u}$ is said to be nonlinearly orbitally stable if $\tilde{u}(\cdot, t)$ approaches $\bar{u}^{\delta(t)}$ as $t \to \infty$, $\bar{u}^\delta$ as defined in (H4), for any solution $\tilde{u}$ of (2) with initial data sufficiently close in some norm to the original profile $\bar{u}$. If, also, the phase $\delta(t)$ converges to a limiting value $\delta(+\infty)$, the profile is said to be nonlinearly phase-asymptotically orbitally stable.*

An important result of [53] was the identification of the following *stability criterion* equivalent to $L^1 \to L^p$ linearized orbital stability of the profile, $p > 1$, where $D(\lambda)$ as described in [14], [53] denotes the Evans function associated with the linearized operator L about the profile: an analytic function analogous to the characteristic polynomial of a finite-dimensional operator, whose zeroes away from the essential spectrum agree in location and multiplicity with the eigenvalues of L.

($\mathcal{D}$) There exist precisely ℓ zeroes of $D(\cdot)$ in the nonstable half-plane Re $\lambda \geq 0$, necessarily at the origin $\lambda = 0$.

As discussed, e.g., in [53], [56]–[58], under assumptions (H0)–(H4), ($\mathcal{D}$) is equivalent to (i) *strong spectral stability*, $\sigma(L) \subset \{\text{Re } \lambda \leq 0\} \cup \{0\}$, (ii) *hyperbolic stability* of the associated ideal shock, and (iii) *transversality* of $\bar{u}$ as a solution of the connection problem in the associated traveling-wave ODE, where hyperbolic stability is defined for Lax and undercompressive shocks by the Lopatinski condition of [33]–[35], [10] and for overcompressive shocks by an analogous long-wave stability condition [56]. Here and elsewhere σ denotes spectrum of a linearized operator or matrix.

The stability condition holds always for small-amplitude Lax profiles [17], [36], [28], [29], [24], [43], [11], but may fail for large-amplitude, or nonclassical over- or undercompressive profiles [1], [14], [12], [60], [56]. It may be readily checked numerically, as described, e.g., in [5]–[6], [8], [9]. It was shown by various techniques in [53], [55], [37]–[41], [44] that the linearized stability condition ($\mathcal{D}$) is also sufficient for nonlinear orbital stability of Lax or overcompressive profiles of arbitrary amplitude. However, up to now, this result had not been verified in the undercompressive case.

In this paper, we present a simple pointwise argument applicable to shocks of any type, establishing that ($\mathcal{D}$) is sufficient for nonlinear phase-asymptotic orbital stability. More precisely, denoting by

$$a_1^\pm < a_2^\pm < \cdots < a_n^\pm \tag{5}$$

the eigenvalues of the limiting convection matrices $A_\pm := df(u_\pm)$, define

$$\theta(x,t) := \sum_{a_j^- < 0} (1+t)^{-1/2} e^{-|x-a_j^- t|^2/Lt} + \sum_{a_j^+ > 0} (1+t)^{-1/2} e^{-|x-a_j^+ t|^2/Lt}, \tag{6}$$

$$\begin{aligned} \psi_1(x,t) := \chi(x,t) &\sum_{a_j^- < 0} (1+|x|+t)^{-1/2}(1+|x-a_j^- t|)^{-1/2} \\ + \chi(x,t) &\sum_{a_j^+ > 0} (1+|x|+t)^{-1/2}(1+|x-a_j^+ t|)^{-1/2}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} \psi_2(x,t) &:= (1-\chi(x,t))(1+|x-a_1^- t| + t^{1/2})^{-3/2} \\ &+ (1-\chi(x,t))(1+|x-a_n^+ t| + t^{1/2})^{-3/2}, \end{aligned} \tag{8}$$

where $\chi(x,t) = 1$ for $x \in [a_1^- t, a_n^+ t]$ and zero otherwise, and $L > 0$ is a sufficiently large constant.

Then, we have the following main theorem.

Theorem 1. *Assuming (H0)–(H4) and the linear stability condition ($\mathcal{D}$), the profile $\bar{u}$ is nonlinearly phase-asymptotically orbitally stable with respect to $C^{0+\alpha}$ initial perturbations $|u_0(x)| \leq E_0(1+|x|)^{-3/2}$, E_0 sufficiently small. More precisely, there exist $\delta(\cdot)$ and $\delta(+\infty)$ such that*

$$\begin{aligned} |\tilde{u}(x,t) - \bar{u}^{\delta(t)}(x)| &\leq CE_0(\theta + \psi_1 + \psi_2)(x,t), \\ |\dot{\delta}(t)| &\leq CE_0(1+t)^{-1}, \\ |\delta(t) - \delta(+\infty)| &\leq CE_0(1+t)^{-1/2}, \end{aligned} \tag{9}$$

where $\tilde{u}$ denotes the solution of (2) *with initial data $\tilde{u}_0 = \bar{u} + u_0$.*

In particular, Theorem 1 yields the desired result of nonlinear stability in the undercompressive or mixed case, effectively completing the one-dimensional stability analysis initiated in [53].

Remark 1. *Pointwise bound* (9) *yields as a corollary the sharp L^p decay rate*

$$|\tilde{u}(x,t) - \bar{u}^{\delta(t)}(x)|_{L^p} \leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}, \quad 1 \leq p \leq \infty. \tag{10}$$

Remark 2. *The profile θ may be recognized as the superposition of Gaussian "approximate diffusion waves" moving along outgoing characteristic directions, while the profiles ψ_1 and ψ_2 respectively account for nonlinear interactions occurring within the characteristic cone $[a_1^- t, a_n^+ t]$ and the algebraically decaying tail of the initial data. The profiles θ and ψ_j correspond roughly to the nonlinear diffusion and linearly coupled waves θ and η*[1] *in the more detailed description of the solution carried out for initial data with the same decay rate in [30], [53] for the Lax and overcompressive case. The latter works estimate $\tilde{u} - \bar{u}$, which contains an additional error*

$$\bar{u}(x) - \bar{u}^{\delta(t)}(x) \sim (\partial \bar{u}^{\delta}/\partial \delta)\delta(t) = O\big(e^{-\eta|x|})(1+t)^{-1/2}\big) \tag{11}$$

near the shock layer that is not present in our analysis.

A difference of the undercompressive from the Lax or overcompressive cases is that the time-asymptotic distribution of mass is no longer determined in a simple way by the mass of the initial data, making difficult the description of nonlinear diffusion and coupled waves. We avoid this difficulty by estimating only joint upper bounds and not the size or shape of component waves. Besides undercompressive stability, this yields also considerable simplification in the pointwise analysis of Lax and overcompressive profiles. In particular, we nowhere attempt to identify cancellation in our estimates of nonlinear interactions, taking into account only transversality of interacting signals, similarly as in (different-family) Glimm interaction estimates for the hyperbolic case [16]. Compare with the analyses of [51, 30, 44] in which a crucial aspect is to identify cancellation in the computation of the linearly coupled wave η. Compensating for the lower resolution in our scheme at the level of diffusion waves is the higher resolution afforded by shock tracking, as reflected in the absence of term (11)*. That is, by sufficiently resolving the nondecaying lowest-order part of the Green function corresponding to shift in the shock location, we are able to ignore the details of higher-order parts.*

Remark 3. *Multidimensional nonlinear $L^1 \cap L^\infty \to L^p$ stability, $p \geq 2$, has been established using $L^q \to L^p$ resolvent bounds for Lax and overcompressive shocks in all dimensions $d \geq 2$ but for undercompressive shocks only in dimensions $d \geq 4$; see [56]–[58]. As discussed in [21], [56], [22], stability of undercompressive shock fronts in physical dimensions $d = 2$ and 3 remains an open question even in the (diffusive-dispersive or diffusive-higher order diffusive) scalar case, for which detailed pointwise Green function bounds are available. An interesting future direction might be to attack this problem by pointwise methods similar to those of this paper.*

Remark 4. *In one dimension, a change of coordinates reduces Slemrod's model for van der Waals gas dynamics or elasticity with viscosity–capillarity to a 2×2 parabolic system with diagonal, strictly parabolic viscosity, to which our results may be applied; see [50, 59]. Likewise, in one dimension, the standard models for imiscible three-phase flow in porous media may be expressed as a 2×2 parabolic system with viscosity that is strictly parabolic on the interior of the physical state space (the set of saturations summing to one) and degenerate on the boundary; see [3]. Thus, our results generically apply here, too. However, note that shocks with one state on the boundary do arise in Riemann problems of physical interest and are not covered by our theory, nor is the degeneracy of the symmetric constant-multiplicity type encountered in real viscosity models. This would be an interesting case for further investigation. In multi-dimensions, neither of these transformations is possible, and so a more special analysis of each specific equation would be required for a stability analysis.*

Plan of the paper. In Section 2, we recall the linearized estimates carried out in [53], [39]. Assuming certain pointwise convolution estimates, we carry out in Section 3 the nonlinear stability analysis of the Lax and undercompressive case. By a slight modification of the argument, we carry out in Section 4 the nonlinear stability analysis of the complementary Lax and overcompressive case. In Section 5, we establish stability of mixed-type shocks in the special case $B \equiv$ constant , and in Section 6 in the general case. Finally, in Section 7, we carry out the deferred convolution estimates, completing the analysis.

[1] Defined in [51] but not explicitly mentioned in either reference, η refers to the contribution of quadratic source terms involving θ alone, i.e., the second iterate in the nonlinear iteration through Duhamel's formula.

2 Linearized estimates

We begin by recalling the pointwise linearized estimates established in [53], [39], expressed in a streamlined form convenient for the nonlinear analysis to follow. Linearizing (2) about a fixed stationary solution $\bar{u}^{\delta_*}(\cdot)$ gives

$$u_t = L^{\delta_*}u := (B^{\delta_*}u_x)_x - (A^{\delta_*}u)_x, \tag{12}$$

where

$$A^{\delta_*}u := df(\bar{u}^{\delta_*}(x))u - dB(\bar{u}^{\delta_*}(x))(u, \bar{u}^{\delta_*}_x), \qquad B^{\delta_*} := B(\bar{u}^{\delta_*}(x)). \tag{13}$$

Proposition 1 ([53], [39]). *Under the assumptions of Theorem 1, the Green function $G(x,t;y)$ associated with the linearized equations* (12) *may be decomposed as $G = E + \tilde{G}$, where*

$$E(x,t;y) = \sum_{j=1}^{\ell} \frac{\partial \bar{u}^{\delta}(x)}{\partial \delta_j}_{|\delta=\delta_*} e_j(y,t), \tag{14}$$

$$e_j(y,t) = \sum_{a_k^->0} \left(errfn\left(\frac{y + a_k^- t}{\sqrt{4\beta_k^- t}}\right) - errfn\left(\frac{y - a_k^- t}{\sqrt{4\beta_k^- t}}\right)\right) l_{jk}^-(y) \tag{15}$$

for $y \le 0$ and symmetrically for $y \ge 0$, with

$$|l_{jk}^{\pm}| \le C, \qquad |(\partial/\partial y) l_{jk}^{\pm}| \le C\gamma e^{-\eta|y|}, \tag{16}$$

and

$$\begin{aligned} &|\partial_{x,y}^{\alpha}\tilde{G}(x,t;y)| \le \\ &\quad C(t^{-|\alpha|/2} + |\alpha_y|\gamma e^{-\eta|y|} + |\alpha_x| e^{-\eta|x|})\Big(\sum_{k=1}^{n} t^{-1/2} e^{-(x-y-a_k^- t)^2/Mt} e^{-\eta x^+} \\ &\quad + \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t| \ge |y|\}} t^{-1/2} e^{-(x - a_j^-(t - |y/a_k^-|))^2/Mt} e^{-\eta x^+}, \\ &\quad + \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t| \ge |y|\}} t^{-1/2} e^{-(x - a_j^+(t - |y/a_k^-|))^2/Mt} e^{-\eta x^-}\Big), \end{aligned} \tag{17}$$

$0 \le |\alpha| \le 2$ *for $y \le 0$ and symmetrically for $y \ge 0$, for some η, C, $M > 0$, where $a_j^{\pm}$ are as in Theorem 1, $\beta_k^{\pm} > 0$, $x^{\pm}$ denotes the positive/negative part of x, indicator function $\chi_{\{|a_k^- t| \ge |y|\}}$ is 1 for $|a_k^- t| \ge |y|$ and 0 otherwise, and $\gamma = 1$ in the mixed or undercompressive case and 0 in the pure Lax or overcompressive case. Moreover, all estimates are uniform in the supressed parameter δ_*.*

Remark 5. *We will refer to the three differently scaled diffusion kernels in (17) respectively as the convection kernel, the reflection kernel, and the transmission kernel. The x-derivative estimate will only be required in the case of mixed-type profiles (see Section 6). Finally, we recall the notation*

$$errfn(z) := \frac{1}{2\pi}\int_{-\infty}^{z} e^{-\xi^2} d\xi.$$

Proof of Proposition 1. This is a restatement of the bounds established in [53], [39] for pure undercompressive, Lax, or overcompressive type profiles; the same argument applies also in the mixed under–overcompressive case. Also, though it was not explicitly stated, uniformity with respect to δ_* is a straightforward consequence of the argument. ∎

Remark 6. *From* (15) *and* (16), *we obtain by straightforward calculation (see [39]) the bounds*

$$\begin{aligned}
|e_j(y,t)| &\le C \sum_{a_k^->0} \left(errfn\left(\frac{y+a_k^-t}{\sqrt{4\beta_k^-t}}\right) - errfn\left(\frac{y-a_k^-t}{\sqrt{4\beta_k^-t}}\right)\right), \\
|e_j(y,t) - e_j(y,+\infty)| &\le C\, errfn(\frac{|y|-at}{M\sqrt{t}}), \quad some\, a>0 \\
|\partial_t e_j(y,t)| &\le Ct^{-1/2} \sum_{a_k^->0} e^{-|y+a_k^-t|^2/Mt}, \\
|\partial_y e_j(y,t)| &\le Ct^{-1/2} \sum_{a_k^->0} e^{-|y+a_k^-t|^2/Mt} \\
&\quad + C\gamma e^{-\eta|y|}\left(errfn\left(\frac{y+a_k^-t}{\sqrt{4\beta_k^-t}}\right) - errfn\left(\frac{y-a_k^-t}{\sqrt{4\beta_k^-t}}\right)\right), \\
|\partial_y e_j(y,t) - \partial_y e_j(y,+\infty)| &\le Ct^{-1/2} \sum_{a_k^->0} e^{-|y+a_k^-t|^2/Mt} \\
|\partial_{yt} e_j(y,t)| &\le C(t^{-1} + \gamma t^{-1/2} e^{-\eta|y|}) \sum_{a_k^->0} e^{-|y+a_k^-t|^2/Mt}
\end{aligned} \tag{18}$$

for $y \le 0$, *and symmetrically for* $y \ge 0$, *where* γ *as above is one for undercompressive profiles and zero otherwise.*

Remark 7. *The main difference between the estimates for the mixed or undercompressive case* $\gamma = 1$ *and the pure Lax or overcompressive case* $\gamma = 0$ *is the presence of slower-decaying* $e^{-\theta|y|}$ *terms in derivative estimates for* e, $\tilde{G}$. *As discussed in [31]–[32], [53], [59], [55], these are not only technical artifacts, but reflect real differences in behavior in the undercompressive case: specifically, that shock dynamics are not governed solely by conservation of mass, as in the Lax or overcompressive case, but by more complicated dynamics of front interaction as indicated by rapidly decaying modes* $\sim e^{-\theta|y|}$.

3 Stability of Lax or undercompressive profiles

We now carry out the proof of Theorem 1 in the Lax or undercompressive case, which may be treated by a particularly simple argument. In these cases $\ell = 1$, and $\bar{u}^\delta = \bar{u}(x-\delta)$, so that we may conveniently work with the "centered" perturbation variable

$$u(x,t) := \tilde{u}(x+\delta(t),t) - \bar{u}(x), \tag{19}$$

for which (2) becomes

$$u_t - Lu = Q(u,u_x)_x + \dot{\delta}(t)(\bar{u}_x + u_x), \tag{20}$$

$L := L^0$, where

$$\begin{aligned}
Q(u,u_x) &= \mathcal{O}(|u|^2 + |u||u_x|) \\
Q(u,u_x)_x &= \mathcal{O}(|u||u_x| + |u_x|^2 + |u||u_{xx}|)
\end{aligned} \tag{21}$$

so long as $|u|$ remains bounded.

Recalling the standard fact that $\bar{u}'$ is a stationary solution of the linearized equations (12), $L\bar{u}' = 0$, or

$$\int_{-\infty}^{\infty} G(x,t;y)\bar{u}_x(y)dy = e^{Lt}\bar{u}_x(x) = \bar{u}'(x),$$

we have by Duhamel's principle:

$$\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} G(x,t;y)u_0(y)\,dy \\
&\quad + \int_0^t \int_{-\infty}^{\infty} G_y(x,t-s;y)(Q(u,u_x) + \dot{\delta}u)(y,s)\,dy\,ds + \delta(t)\bar{u}'(x).
\end{aligned}$$

Defining

$$\begin{aligned}\delta(t) = &-\int_{-\infty}^{\infty} e(y,t)u_0(y)\,dy \\ &+\int_0^t\int_{-\infty}^{+\infty} e_y(y,t-s)(Q(u,u_x)+\dot{\delta}\,u)(y,s)dyds,\end{aligned} \tag{22}$$

following [53], [55], [37]–[38], where e is defined as in (15) (that is, $e=\sum_j e_j$), and recalling the decomposition $G=E+\tilde{G}$, we obtain finally the *reduced equations*:

$$\begin{aligned}u(x,t) = &\int_{-\infty}^{\infty} \tilde{G}(x,t;y)u_0(y)\,dy \\ &-\int_0^t\int_{-\infty}^{\infty} \tilde{G}_y(x,t-s;y)(Q(u,u_x)+\dot{\delta}u)(y,s)dy\,ds,\end{aligned} \tag{23}$$

and, differentiating (22) with respect to t, and observing that $e_y(y,s)\rightharpoonup 0$ as $s\to 0$, as the difference of approaching heat kernels:

$$\begin{aligned}\dot{\delta}(t) = &-\int_{-\infty}^{\infty} e_t(y,t)u_0(y)\,dy \\ &+\int_0^t\int_{-\infty}^{+\infty} e_{yt}(y,t-s)(Q(u,u_x)+\dot{\delta}u)(y,s)\,dy\,ds.\end{aligned} \tag{24}$$

We shall make use of the following three technical lemmas, the proofs of which are given in Section 7.

Lemma 1 (Short-time theory). *Under the assumptions of Theorem 1, for data* $u_0\in C^{0+\alpha}(x)$*, equations* (22)–(23) *(alternatively,* (50), (41) *of the following section) admit a unique local solution* $u\in C^{0+\alpha}(x)\cap C^{0+\alpha/2}(t)$, $\delta\in C^{1+\alpha/2}(t)$*, extending so long as* $|u|_{C^{0+\alpha}}$ *remains bounded. Moreover, on this domain,* $\sup_z |u|(\theta+\psi_1+\psi_2)^{-1}(z,\cdot)$ *remains continuous so long as it and* $|\dot{\delta}(1+t)|$ *are uniformly bounded and, for* $t\ge\tau>0$ *sufficiently small,*

$$\sup_z |u_x|(\theta+\psi_1+\psi_2)^{-1}(z,t)\le C\tau^{-1/2}\sup_z |u|(\theta+\psi_1+\psi_2)^{-1}(z,t-\tau). \tag{25}$$

Lemma 2 (Linear estimates). *Under the assumptions of Theorem 1,*

$$\begin{aligned}\int_{-\infty}^{+\infty}|\tilde{G}(x,t;y)|(1+|y|)^{-3/2}\,dy &\le C(\theta+\psi_1+\psi_2)(x,t),\\ \int_{-\infty}^{+\infty}|e_t(y,t)|(1+|y|)^{-3/2}\,dy &\le C(1+t)^{-3/2},\\ \int_{-\infty}^{+\infty}|e(y,t)|(1+|y|)^{-3/2}\,dy &\le C,\\ \int_{-\infty}^{+\infty}|e(y,t)-e(y,+\infty)|(1+|y|)^{-3/2}\,dy &\le C(1+t)^{-1/2},\end{aligned} \tag{26}$$

for $0\le t\le+\infty$*, any* a, $M>0$*, for some* $C>0$*, where* $\tilde{G}$ *and* e *are defined as in Proposition 1.*

Lemma 3 (Nonlinear estimates). *Under the assumptions of Theorem 1,*

$$\begin{aligned}\int_0^t\int_{-\infty}^{+\infty}|\tilde{G}_y(x,t-s;y)|\Psi(y,s)\,dyds &\le C(\theta+\psi_1+\psi_2)(x,t),\\ \int_0^t\int_{-\infty}^{+\infty}|e_{yt}(y,t-s)|\Psi(y,s)\,dyds &\le C(1+t)^{-1},\\ \int_0^{+\infty}\int_{-\infty}^{+\infty}|e_y(y,+\infty)|\Psi(y,s)\,dy\,ds &\le C\gamma,\\ \int_0^t\int_{-\infty}^{+\infty}|e_y(y,t-s)-e_y(y,+\infty)|\Psi(y,s)\,dyds &\le C(1+t)^{-1/2},\\ \int_t^{+\infty}\int_{-\infty}^{+\infty}|e_y(y,t-s)|\Psi(y,s)\,dy &\le C(1+t)^{-1/2},\end{aligned}\tag{27}$$

for

$$\begin{aligned}\Psi(y,s) := (1+s)^{1/2}s^{-1/2}(\theta+\psi_1+\psi_2)^2(y,s)\\ +(1+s)^{-1}(\theta+\psi_1+\psi_2)(y,s)\end{aligned}\tag{28}$$

Proof of Theorem 1, Lax or undercompressive case. Define

$$\zeta(t) := \sup_{y,0\le s\le t}\Big(|u|(\theta+\psi_1+\psi_2)^{-1}(y,t)+|\dot{\delta}(s)|(1+s)\Big).\tag{29}$$

We shall establish:

Claim. For all $t\ge 0$ for which a solution exists with ζ uniformly bounded by some fixed, sufficiently small constant, there holds

$$\zeta(t)\le C_2(E_0+\zeta(t)^2).\tag{30}$$

From this result, provided $E_0<1/4C_2$, we have that $\zeta(t)\le 2C_2E_0$ implies $\zeta(t)<2C_2E_0$, and so we may conclude by continuous induction that

$$\zeta(t)<2C_2E_0\tag{31}$$

for all $t\ge 0$. (By Lemma 1, $u\in C^1$ exists and ζ remains continuous so long as ζ remains bounded by some uniform constant, hence (31) is an open condition.) Thus, it remains only to establish the claim above.

Proof of Claim. We must show that $u(\theta+\psi_1+\psi_2)^{-1}$ and $|\dot{\delta}(s)|(1+s)$ are each bounded by $C(E_0+\zeta(t)^2)$, for some $C>0$, all $0\le s\le t$, so long as ζ remains sufficiently small.

By (29), combined with (25), we have for $t\ge 1$ that

$$\begin{aligned}|u_x(x,t)| &\le C\zeta(t-1)(\theta+\psi_1+\psi_2)(x,t-1)\\ &\le C_2\zeta(t)(\theta+\psi_1+\psi_2)(x,t)\end{aligned}\tag{32}$$

and for $0\le t\le 1$ that

$$\begin{aligned}|u_x(x,t)| &\le Ct^{-1/2}\zeta(0)(\theta+\psi_1+\psi_2)(x,0)\\ &\le C_2\zeta(t)t^{-1/2}(\theta+\psi_1+\psi_2)(x,t).\end{aligned}\tag{33}$$

Combining these estimates, and recalling definition (29), we obtain for all $t\ge 0$ and some $C>0$ that

$$\begin{aligned}|\dot{\delta}(t)| &\le \zeta(t)(1+t)^{-1},\\ |u(x,t)| &\le \zeta(t)(\theta+\psi_1+\psi_2)(x,t),\\ |u_x(x,t)| &\le C\zeta(t)(1+t)^{1/2}t^{-1/2}(\theta+\psi_1+\psi_2)(x,t)\end{aligned}\tag{34}$$

and therefore

$$|(Q(u,u_x)+\dot{\delta}u)(y,s)|\le C\Psi(y,s)\tag{35}$$

with Ψ as defined in (28).

Combining (35) with representations (23)–(24) and applying Lemmas 2 and 3, we obtain

$$\begin{aligned}
|u(x,t)| &\le \int_{-\infty}^{\infty} |\tilde{G}(x,t;y)||u_0(y)|\,dy \\
&\quad + \int_0^t \int_{-\infty}^{\infty} |\tilde{G}_y(x,t-s;y)||(Q(u,u_x)+\dot{\delta}u)(y,s)|dy\,ds \\
&\le E_0 \int_{-\infty}^{\infty} |\tilde{G}(x,t;y)|(1+|y|)^{-3/2}\,dy \\
&\quad + C\zeta(t)^2 \int_0^t \int_{-\infty}^{\infty} |\tilde{G}_y(x,t-s;y)|\Psi(y,s)dy\,ds \\
&\le C(E_0+\zeta(t)^2)(\theta+\psi_1+\psi_2)(x,t)
\end{aligned}$$

and, similarly,

$$\begin{aligned}
|\dot{\delta}(t)| &\le \int_{-\infty}^{\infty} |e_t(y,t)||u_0(y)|\,dy \\
&\quad + \int_0^t \int_{-\infty}^{+\infty} |e_{yt}(y,t-s)||(Q(u,u_x)+\dot{\delta}u)(y,s)|\,dy\,ds \\
&\le \int_{-\infty}^{\infty} |e_t(y,t)|(1+|y|)^{-3/2}\,dy + \int_0^t \int_{-\infty}^{+\infty} |e_{yt}(y,t-s)|\Psi(y,s)\,dy\,ds \\
&\le C(E_0+\zeta(t)^2)(1+t)^{-1}.
\end{aligned}$$

Dividing by $(\theta+\psi_1+\psi_2)(x,t)$ and $(1+t)^{-1}$, respectively, we obtain (30) as claimed.

From (30), we obtain global existence, with $\zeta(t) \le 2CE_0$. From the latter bound and the definition of ζ in (29) we obtain the first two bounds of (9). It remains to establish the third bound, expressing convergence of phase δ to a limiting value $\delta(+\infty)$.

By Lemmas 2–3 together with the previously obtained bounds (35) and $\zeta \le CE_0$, and the definition (29) of ζ, the formal limit

$$\begin{aligned}
\delta(+\infty) &:= \int_{-\infty}^{\infty} e(y,+\infty)u_0(y)\,dy \\
&\quad + \int_0^{+\infty} \int_{-\infty}^{+\infty} e_y(y,+\infty)(Q(u,u_x)+\dot{\delta}u)(y,s)\,dy\,ds \\
&\le \int_{-\infty}^{\infty} E_0|e(y,+\infty)|(1+|y|)^{-3/2}\,dy \\
&\quad + \int_0^{+\infty} \int_{-\infty}^{+\infty} CE_0|e_y(y,+\infty)|\Psi(y,s)\,dy\,ds \\
&\le CE_0
\end{aligned}$$

is well-defined, as the sum of absolutely convergent integrals.

Applying Lemmas 2–3 a final time, we obtain

$$\begin{aligned}|\delta(t)-\delta(+\infty)| &\le \int_{-\infty}^{\infty}|e(y,t)-e(y,+\infty)||u_0(y)|\,dy\\ &\quad+\int_0^t\int_{-\infty}^{+\infty}|e_y(y,t-s)-e_y(y,+\infty)||(Q(u,u_x)+\dot\delta u)(y,s)|\,dy\,ds\\ &\quad+\int_t^{+\infty}\int_{-\infty}^{+\infty}|e_y(y,+\infty)||(Q(u,u_x)+\dot\delta u)(y,s)|\,dy\,ds\\ &\le \int_{-\infty}^{\infty}|e(y,t)-e(y,+\infty)|(1+|y|)^{-3/2}\,dy\\ &\quad+\int_0^t\int_{-\infty}^{+\infty}|e_y(y,t-s)-e_y(y,+\infty)|\Psi(y,s)\,dy\,ds\\ &\quad+\int_t^{+\infty}\int_{-\infty}^{+\infty}|e_y(y,+\infty)|\Psi(y,s)\,dy\,ds\\ &\le CE_0(1+t)^{-1/2},\end{aligned}$$

establishing the remaining bound and completing the proof. ∎

4 Overcompressive profiles.

We may treat the overcompressive case by a slight modification of the argument of Section 3, which applies also to the Lax case. As the Lax and overcompressive case have already been treated by different means in [53], [44] we shall only sketch the changes necessary for the argument, omitting most details.

Modified equations. In the overcompressive case, $\ell > 1$, $\bar u^\delta$ consists not only of translates of $\bar u$, but also of orbits distinct from $\bar u$. In particular, the different representatives are not all derived from a group action, and so we cannot use a centering transformation as in (19), which consists of the group operations

$$T_\delta(\tilde u - T_{-\delta}\bar u) = T_\delta \tilde u - \bar u,$$

where $T_\alpha v(x,t) := v(x+\alpha,t)$ denotes translation in x. Accordingly, we work with the primitive variable

$$u(x,t) := \tilde u(x,t) - \bar u^{\delta(t)}(x) \tag{36}$$

and center the *equations* instead, about some strategically chosen δ_*, obtaining in place of (20) the modified perturbation equation

$$u_t - L^{\delta_*}u = Q^{\delta_*}(u,u_x)_x + \dot\delta(t)(\partial\bar u^\delta/\partial\delta)_{|\delta_*} + R^{\delta_*}(\delta,u,u_x)_x + S^{\delta_*}(\delta,\delta_t), \tag{37}$$

where Q^{δ_*} is as in (21) and

$$\begin{aligned}R^{\delta_*} &= \Big(A(\bar u^{\delta_*}(x)) - A(\bar u^{\delta(t)}(x))\Big)u + \Big(B(\bar u^{\delta_*}(x)) - B(\bar u^{\delta(t)}(x))\Big)u_x\\ &= \mathcal{O}(e^{-\eta|y|}|\delta-\delta_*|(|u|+|u_x|))\end{aligned} \tag{38}$$

and

$$S^{\delta_*} = \dot\delta\Big((\partial\bar u^\delta/\partial\delta)_{|\delta(t)} - (\partial\bar u^\delta/\partial\delta)_{|\delta_*}\Big) = \mathcal{O}(e^{-\eta|y|}|\dot\delta||\delta-\delta_*|) \tag{39}$$

account for "centering errors"; see, e.g., [19], [21], [57] for related computations.

Defining

$$\begin{aligned}\delta(t) &:= \delta_* - \int_{-\infty}^{\infty} e(y,t)u_0^{\delta_*}(y)\,dy - \int_0^t\int_{-\infty}^{+\infty} e(y,t-s)S^{\delta_*}(\delta,\delta_t)(y,s)dyds\\ &\quad+\int_0^t\int_{-\infty}^{+\infty} e_y(y,t-s)(Q^{\delta_*}(u,u_x)+R^{\delta_*}(\delta,u,u_x))(y,s)dyds,\\ u_0^{\delta_*} &:= \tilde u_0 - \bar u^{\delta_*}\end{aligned} \tag{40}$$

by analogy with (22), we obtain reduced equations

$$\begin{aligned}u(x,t) &= \int_{-\infty}^{\infty} \tilde G(x,t;y)u_0^{\delta_*}(y)\,dy\\ &\quad-\int_0^t\int_{-\infty}^{\infty} \tilde G(x,t-s;y)S^{\delta_*}(\delta,\delta_t))(y,s)dy\,ds\\ &\quad-\int_0^t\int_{-\infty}^{\infty} \tilde G_y(x,t-s;y)(Q^{\delta_*}(u,u_x)+R^{\delta_*}(\delta,u,u_x))(y,s)dy\,ds\end{aligned} \tag{41}$$

and

$$\begin{aligned}\dot\delta(t) &= -\int_{-\infty}^{\infty} e_t(y,t)u_0^{\delta_*}(y)\,dy\\ &\quad-\int_0^t\int_{-\infty}^{+\infty} e_t(y,t-s)S^{\delta_*}(\delta,\delta_t)(y,s)\,dy\,ds\\ &\quad+\int_0^t\int_{-\infty}^{+\infty} e_{yt}(y,t-s)(Q^{\delta_*}(u,u_x)+R^{\delta_*}(\delta,u,u_x))(y,s)\,dy\,ds.\end{aligned} \tag{42}$$

Asymptotic shock location. An important feature of the Lax or overcompressive case is that the stability criterion $(\mathcal{D})$ in these cases implies that the (nonlinear) L^1-asymptotic state of the perturbed shock must be formally determined by conservation of mass, in the sense that relation

$$\begin{aligned}\int_{-\infty}^{+\infty}(\tilde u_0(y)-\bar u(y))dy &= \int(\bar u^{\delta_{+\infty}}(y)-\bar u(y))dy\\ &\quad+\sum_{a_j^+>0} m_j^+r_j^+ + \sum_{a_j^-<0} m_j^-r_j^-\end{aligned} \tag{43}$$

is full rank, hence uniquely soluble for $m_j^\pm$, $\delta_{+\infty}$ for $\int_{-\infty}^{+\infty}(\tilde u-\bar u)(y)dy$ sufficiently small, where $a_j^\pm$ and $r_j^\pm$ denote eigenvalues and right eigenvectors of $A_\pm = f'(u_\pm)$, $m_j^\pm$ denotes asymptotic mass in the jth outgoing characteristic field at $\pm\infty$, and $\delta_{+\infty}$ denotes the asymptotic shock location, or, equivalently,

$$\int_{-\infty}^{+\infty}(\tilde u_0-\bar u^{\delta_{+\infty}})(y)\,dy = \sum_{a_j^\pm\gtrless 0} m_j^\pm r_j^\pm \tag{44}$$

for $\delta_{+\infty}=\mathcal{O}(E_0)$, the latter estimate a consequence of full rank; for further discussion, see [31]–[32], [53], [56], and references therein. Centering about $\delta_*=\delta_{+\infty}$, we may thus arrange that

$$\int_{-\infty}^{+\infty} u_0^{\delta_*}(y) = \int_{-\infty}^{+\infty}(\tilde u_0-\bar u^{\delta_*})(y)\,dy = \sum_{a_j^\pm\gtrless 0} m_j^\pm r_j^\pm, \tag{45}$$

while maintaining our assumptions on initial perturbation $u_0^{\delta_*}$. In these coordinates, we may expect that $|\delta(t)-\delta_*|$ decays to zero.

A second consequence of (43), this time at the linearized level, is that $e(y, +\infty)$ (constant in the Lax or overcompressive case; see Proposition 1) must be orthogonal to all "outgoing modes" r_j^+, $a_j^+ > 0$ and and r_j^-, $a_j^- < 0$, hence with choice of coordinates (45) we have

$$\int_{-\infty}^{+\infty} e(y, +\infty) u_0^{\delta_*}(y)\, dy = 0. \tag{46}$$

A final consequence of (43) is that the map

$$\delta \to \hat{\delta} := \int_{-\infty}^{+\infty} \Pi(\bar{u}^\delta - \bar{u})(y)\, dy \in \mathbb{R}^\ell \tag{47}$$

must be invertible, where $\Pi \in \ell \times n$ is any (constant) full rank matrix with rows orthogonal to outgoing modes $r_j^\pm$. Reparametrizing by $\delta = \hat{\delta}$, we may thus arrange that $\delta = \int_{-\infty}^{+\infty} \Pi(\bar{u}^\delta - \bar{u})(y)\, dy$, and thus

$$\int_{-\infty}^{+\infty} \Pi(\partial \bar{u}^\delta / \delta)(y)\, dy \equiv I_\ell \tag{48}$$

for δ in a neighborhood of the origin, so that $\int_{-\infty}^{+\infty} \Pi S^{\delta_*}(y, s)\, dy \equiv 0$ for all s, and therefore

$$\int_{-\infty}^{+\infty} e(y, +\infty) S^{\delta_*}(y, s)\, dy \equiv 0. \tag{49}$$

Combining (46) and (49), we obtain the alternative representation

$$\begin{aligned} \delta(t) = \delta_* &- \int_{-\infty}^{\infty} (e(y,t) - e(y,+\infty)) u_0(y)\, dy \\ &- \int_0^t \int_{-\infty}^{+\infty} (e(y, t-s) - e(y, +\infty)) S^{\delta_*}(\delta, \delta_t)(y,s) dy ds \\ &+ \int_0^t \int_{-\infty}^{+\infty} e_y(y, t-s)(Q^{\delta_*}(u, u_x) + R^{\delta_*}(\delta, u, u_x))(y,s) dy ds, \end{aligned} \tag{50}$$

from which we may observe decay in $|\delta - \delta_*|$ *without a priori knowledge of the global behavior of* u.

Stability argument. Working within the framework of equations (41), (42), and (50), we can carry out the proof of Theorem 1 for the Lax or overcompressive case by essentially the same argument presented for the Lax and undercompressive case in the previous section using Lemmas 1–3 together with the following Lemma proved in Section 7.

Lemma 4 (Auxiliary estimates). *Under the assumptions of Theorem 1,*

$$\begin{aligned} \int_0^t \int_{-\infty}^{+\infty} |\tilde{G}_y(x, t-s; y)| \Phi_1(y,s)\, dy ds &\le C(\theta + \psi_1 + \psi_2)(x,t), \\ \int_0^t \int_{-\infty}^{+\infty} |e_{yt}(y, t-s)| \Phi_1(y,s)\, dy ds &\le C(1+t)^{-1}, \\ \int_0^t \int_{-\infty}^{+\infty} |e_y(y, t-s)| \Phi_1(y,s)\, dy ds &\le C(1+t)^{-1/2} \end{aligned} \tag{51}$$

and

$$\begin{aligned} \int_0^t \int_{-\infty}^{+\infty} |\tilde{G}(x, t-s; y)| \Phi_2(y,s)\, dy ds &\le C(\theta + \psi_1 + \psi_2)(x,t), \\ \int_0^t \int_{-\infty}^{+\infty} |e_t(y, t-s)| \Phi_2(y,s)\, dy ds &\le C(1+t)^{-3/2}, \\ \int_0^t \int_{-\infty}^{+\infty} |e(y, t-s) - e(y, +\infty)| \Phi_2(y,s)\, dy ds &\le C(1+t)^{-3/2}, \end{aligned} \tag{52}$$

where

$$\begin{aligned}\Phi_1(y,s) &:= e^{-\eta|y|}s^{-1/2}(\theta+\psi_1+\psi_2)(y,s) \le Ce^{-\eta|y|/2}s^{-1/2}(1+s)^{-1},\\ \Phi_2(y,s) &:= e^{-\eta|y|}(1+s)^{-3/2}.\end{aligned} \tag{53}$$

Specifically, defining

$$\zeta(t) := \sup_{y,0\le s\le t}\Big(|u|(\theta+\psi_1+\psi_2)^{-1}(y,t) + |\dot\delta(s)|(1+s) + |\delta(s)-\delta_*|(1+s)^{1/2}\Big), \tag{54}$$

noting that

$$|Q^{\delta_*}| \le \zeta^2\Psi, \quad |R^{\delta_*}| \le \zeta^2(\Psi+\Phi_1), \quad |S^{\delta_*}| \le \zeta^2\Phi_2,$$

and applying our convolution lemmas, we may obtain (30) as before, yielding at once global existence and the claimed rates of decay.

Remark 8. *It was crucial in the argument to linearize about the limiting profile $\bar u^{\delta_{+\infty}}$ in order that error S be manageable, i.e., $\delta \to \delta_*$ as $t \to +\infty$.*

5 Mixed type profiles, constant viscosity case.

We now present an alternative proof subsuming Lax, undercompressive, overcompressive, and even mixed under–overcompressive cases in a single argument. For clarity of exposition, we first restrict to the simpler case $B \equiv$ constant , which permits also the following slightly stronger result. The general case is treated in Section 6.

Theorem 2. *Let $B \equiv$ constant . Then, assuming (H0)–(H4), and stability condition ($\mathcal{D}$), the profile $\bar u$ is nonlinearly phase-asymptotically orbitally stable with respect to (not necessarly Hölder continuous) initial perturbations $|u_0(x)| \le E_0(1+|x|)^{-3/2}$, E_0 sufficiently small. More precisely, (9) is satisfied for some $\delta(\cdot)$, $\delta(+\infty)$, where $\tilde u$ denotes the solution of (2) with initial data $\tilde u_0 = \bar u + u_0$.*

Proof. Defining u as in (36), we obtain

$$u_t - L^{\delta_*}u = Q^{\delta_*}(u)_x + \dot\delta(t)(\partial\bar u^\delta/\partial\delta)_{|\delta_*} + R^{\delta_*}(\delta,u)_x + S^{\delta_*}(\delta,\delta_t), \tag{55}$$

where

$$Q^{\delta_*} = \mathcal{O}(|u|^2), \tag{56}$$

$$R^{\delta_*} = \Big(A(\bar u^{\delta_*}(x)) - A(\bar u^{\delta(t)}(x))\Big)u = \mathcal{O}(e^{-\eta|y|}|\delta-\delta_*||u|) \tag{57}$$

and S^{δ_*} is as in (39). Defining

$$\begin{aligned}\delta(t) = \delta_* &- \int_{-\infty}^{\infty} e(y,t)u_0^{\delta_*}(y)\,dy - \int_0^t\int_{-\infty}^{+\infty} e(y,t-s)S^{\delta_*}(\delta,\delta_t)(y,s)dyds\\ &+ \int_0^t\int_{-\infty}^{+\infty} e_y(y,t-s)(Q^{\delta_*}(u)+R^{\delta_*}(\delta,u))(y,s)dyds,\\ u_0^{\delta_*}(y) &:= (\tilde u - \bar u^{\delta_*})(y),\end{aligned} \tag{58}$$

we obtain

$$\begin{aligned}u(x,t) = \mathcal{T}_u(u,\delta,\dot\delta,\delta_*)(t) := &\int_{-\infty}^{\infty}\tilde G(x,t;y)\bar u_0^{\delta_*}(y)\,dy\\ &- \int_0^t\int_{-\infty}^{\infty}\tilde G(x,t-s;y)S^{\delta_*}(\delta,\delta_t))(y,s)dy\,ds\\ &- \int_0^t\int_{-\infty}^{\infty}\tilde G_y(x,t-s;y)(Q^{\delta_*}(u)+R^{\delta_*}(\delta,u))(y,s)dy\,ds\end{aligned} \tag{59}$$

and

$$\begin{aligned}\dot\delta(t) = \mathcal{T}_{\dot\delta}(u,\delta,\dot\delta,\delta_*)(t) :=& -\int_{-\infty}^{\infty} e_t(y,t)\bar u_0^{\delta_*}(y)\,dy\\ &-\int_0^t\int_{-\infty}^{+\infty} e_t(y,t-s)S^{\delta_*}(\delta,\delta_t)(y,s)\,dy\,ds\\ &+\int_0^t\int_{-\infty}^{+\infty} e_{yt}(y,t-s)(Q^{\delta_*}(u)+R^{\delta_*}(\delta,u))(y,s)\,dy\,ds.\end{aligned} \tag{60}$$

Defining now

$$\begin{aligned}&\mathcal{T}_\delta(u,\delta,\dot\delta,\delta_*)(t) :=\\ &-\int_{-\infty}^{\infty}(e(y,t)-e(y,+\infty))\bar u_0^{\delta_*}(y)\,dy\\ &\quad-\int_0^t\int_{-\infty}^{+\infty}(e(y,t-s)-e(y,+\infty))S^{\delta_*}(\delta,\dot\delta)(y,s)\,dy\,ds\\ &\quad+\int_0^t\int_{-\infty}^{+\infty}(e_y(y,t-s)-e_y(y,+\infty))(Q^{\delta_*}(u)+R^{\delta_*}(\delta,u))(y,s)\,dy\,ds\\ &\quad+\int_t^{+\infty}\int_{-\infty}^{+\infty} e(y,+\infty)S^{\delta_*}(\delta,\dot\delta)(y,s)\,dy\,ds,\\ &\quad-\int_t^{+\infty}\int_{-\infty}^{+\infty} e_y(y,+\infty)((Q^{\delta_*}(u)+R^{\delta_*}(\delta,u))(y,s)\,dy\,ds,\end{aligned} \tag{61}$$

and

$$\begin{aligned}&\mathcal{T}_{\delta_{+\infty}}(u,\delta,\dot\delta,\delta_*)(t) :=\\ &\quad\delta_* - \int_{-\infty}^{\infty} e(y,+\infty))\bar u_0^{\delta_*}(y)\,dy\\ &\qquad-\int_0^{+\infty}\int_{-\infty}^{+\infty} e(y,+\infty)S^{\delta_*}(\delta,\dot\delta)(y,s)\,dy\,ds\\ &\qquad+\int_0^{+\infty}\int_{-\infty}^{+\infty} e_y(y,+\infty)(Q^{\delta_*}(u)+R^{\delta_*}(\delta,u))(y,s)\,dy\,ds,\end{aligned} \tag{62}$$

we may express the solution of (58)–(60) equivalently as the solution of the fixed-point equation

$$(u,\delta,\dot\delta) = (\mathcal{T}_u,\mathcal{T}_\delta,\mathcal{T}_{\dot\delta})(u,\delta,\dot\delta,\delta_*). \tag{63}$$

in combination with

$$\mathcal{T}_{\delta_{+\infty}}(u,\delta,\dot\delta,\delta_*) = \delta_*. \tag{64}$$

Defining norm

$$\begin{aligned}|(f,g,h)|_\zeta :=& |f(\theta+\psi_1+\psi_2)^{-1}|_{L^\infty(x,t)} + |g(t)(1+t)^{1/2}|_{L^\infty(t)}\\ &+|h(t)(1+t)|_{L^\infty(t)}\end{aligned} \tag{65}$$

and Banach space

$$\mathcal{B} := \{(f,g,h):\, |f,g,h|_\zeta < +\infty\}, \tag{66}$$

we find by the estimates of the previous sections that, for

$$|\tilde u_0 - \bar u|(x) \le E_0(1+|x|)^{-3/2},$$

E_0 sufficiently small, $(\mathcal{T}_u,\mathcal{T}_\delta,\mathcal{T}_{\dot\delta},\mathcal{T}_{\delta_{+\infty}})$ is a well-defined mapping from

$$B(0,r)\subset \mathcal{B}\times\mathbb{R}\to\mathcal{B}\times\mathbb{R}$$

for $r>0$ sufficiently small, with

$$|\mathcal{T}|_{\mathcal{B}\times\mathbb{R}} = \mathcal{O}(E_0 + |\delta_*| + |(u,\delta,\dot\delta)|_\zeta^2). \tag{67}$$

Moreover, essentially the same estimates yield that $\mathcal{T}$ is Frechet differentiable on $B(0,r)$, with

$$\frac{\partial(\mathcal{T}_u, \mathcal{T}_\delta, \mathcal{T}_{\dot\delta}, \mathcal{T}_{\delta_{+\infty}})}{\partial(u, \delta, \dot\delta)} = \mathcal{O}(|(u, \delta, \dot\delta|_\zeta), \tag{68}$$

$$\frac{\partial(\mathcal{T}_u, \mathcal{T}_\delta, \mathcal{T}_{\dot\delta})}{\partial\delta_*} = \mathcal{O}(1), \tag{69}$$

and

$$\frac{\partial\mathcal{T}_{\delta_{+\infty}}}{\partial\delta_*} = \mathcal{O}(|(u, \delta, \dot\delta|_\zeta), \tag{70}$$

where the final equality (70) follows from

$$\begin{aligned}(\partial(\mathcal{T}_{\delta_{+\infty}} - \delta_*)/\partial\delta_*) &= \int_{-\infty}^{\infty} e(y, +\infty))(\partial\bar{u}^\delta/\partial\delta)_{|\delta_*}(y)\,dy + \mathcal{O}(|(u, \delta, \dot\delta|_\zeta)\\ &= I_\ell + \mathcal{O}(|(u, \delta, \dot\delta|_\zeta).\end{aligned} \tag{71}$$

In turn, relation

$$\int_{-\infty}^{\infty} e(y, +\infty))(\partial\bar{u}^\delta/\partial\delta)_{|\delta_*}(y)\,dy = I_\ell$$

follows from the standard fact that $L^{\delta_*}(\partial\bar{u}^\delta/\partial\delta)_{|\delta_*} = 0$, hence

$$\int_{-\infty}^{+\infty} G(x,t;y)(\partial\bar{u}^\delta/\partial\delta)_{|\delta_*}(y)\,dy \equiv (\partial\bar{u}^\delta/\partial\delta)_{|\delta_*}(x)$$

which, together with the fact that $E = (\partial\bar{u}^\delta/\partial\delta)_{|\delta_*}(x)e(y,t))$ represents the only nondecaying part of $G(x,t;y)$ under stability criterion $(\mathcal{D})$, yields

$$(\partial\bar{u}^\delta/\partial\delta)_{|\delta_*}(x)\int_{-\infty}^{+\infty} e(x, +\infty; y)(\partial\bar{u}^\delta/\partial\delta)_{|\delta_*}(y)\,dy = (\partial\bar{u}^\delta/\partial\delta)_{|\delta_*}(x)$$

in the limit as $t \to +\infty$.

Combining (68)–(70), we find that, for E_0, r sufficiently small, $\mathcal{T}$ is contractive with respect to norm

$$|(i,j,k,l)|_* := |(i,j,k)|_\zeta + C|l|, \tag{72}$$

for $C > 0$ sufficiently large, with $|\mathcal{T}(0)|_* = \mathcal{O}(E_0)$. Applying the Contraction Mapping Theorem, we find that (63)–(64) have a unique solution in $\mathcal{B}$, from which the stated decay estimates follow by definition of $|\cdot|_\zeta$. ∎

Remark 9. *The above contraction mapping argument may be recognized as an alternative version of the Implicit Function Theorem argument commonly used to establish orbital stability, as for example in [20], [45].*

6 Mixed type profiles, general case.

The constant-viscosity assumption of the previous section made it possible to work in a weighted L^∞ norm, since we needed to gain only a single derivative in the associated nonlinear iteration scheme. To treat the general case, we work instead in a Hölder space using Schauder-type smoothing estimates like those of Lemma 1.

Proof of Theorem 1, general case. In the general, variable-viscosity case, we may express the perturbation u of (36) again as the solution of equation (55), but with Q^{δ_*} and R^{δ_*} now depending also on u_x.

To accomodate this fact, we impose on u_0 also Hölder continuity,

$$|u_0|_{C^{0+\alpha}} \le C, \tag{73}$$

and impose on u the uniform bounds

$$|u|_{C^{0+\alpha}} \le C, \qquad |u|_{C^1} \le Ct^{-\frac{1}{2}+\alpha}, \qquad |u|_{C^2} \le Ct^{-1+\alpha}, \tag{74}$$

or equivalently $|u|_\alpha \le C$, for

$$|u|_\alpha := \sup_{s\ge 0}\Big(|u|_{C^{0+\alpha}} + |u|_{C^1}s^{\frac{1}{2}-\alpha} + |u|_{C^2}s^{1-\alpha}\Big), \tag{75}$$

together with $|(u,\delta,\dot\delta)|_{\zeta_1}$, $|\delta_*| \le E_0$ sufficiently small, for

$$\begin{aligned}|(f,g,h)|_{\zeta_1} :=\ & |f(\theta+\psi_1+\psi_2)^{-1}|_{L^\infty(x,t)}\\ &+ |\partial_x f t^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}(\theta+\psi_1+\psi_2)^{-1}|_{L^\infty(x,t)}\\ &+ |g(t)(1+t)^{1/2}|_{L^\infty(t)} + |h(t)(1+t)|_{L^\infty(t)}\end{aligned} \tag{76}$$

(note: now augmented with derivative bound). Denote

$$\begin{aligned}\mathcal{B}_1 &:= \{(f,g,h) :\ |f,g,h|_{\zeta_1} < +\infty\}\\ \mathcal{C} &:= \{(f,g,h) :\ |f|_\alpha < +\infty\}.\end{aligned} \tag{77}$$

We now define $\mathcal{T}_\delta$, $\mathcal{T}_{\dot\delta}$, and $\mathcal{T}_{\delta_\infty}$ by (60)–(62), exactly as before. However, we define $\mathcal{T}_u$ now implicitly, as the solution of

$$\begin{aligned}\mathcal{T}_u(u,\delta,\dot\delta,\delta_*)(t) = &\int_{-\infty}^{\infty}\tilde G(x,t;y)\bar u_0^{\delta_*}(y)\,dy\\ &-\int_0^t\int_{-\infty}^{\infty}\tilde G(x,t-s;y)S^{\delta_*}(\delta,\delta_t))(y,s)dy\,ds\\ &-\int_0^t\int_{-\infty}^{\infty}\tilde G_y(x,t-s;y)(Q^{\delta_*}(u,\partial_x\mathcal{T}_u)\\ &+R^{\delta_*}(\delta,u,\partial_x\mathcal{T}_u))(y,s)dy\,ds,\end{aligned} \tag{78}$$

or equivalently as the solution v of

$$\begin{aligned}v_t - \hat L^{\delta_*}(u)v = &\hat Q^{\delta_*}(u)_x + \dot\delta(t)(\partial\bar u^\delta/\partial\delta)_{|\delta_*} + \hat R^{\delta_*}(\delta,u)_x\\ &+ S^{\delta_*}(\delta,\delta_t), + \hat T^{\delta_*}(u,u_x,v_x),\end{aligned} \tag{79}$$

where

$$\hat L^{\delta_*}(u)v := B(\bar u^{\delta_*}+u)v_{xx} - (A(\bar u^{\delta_*})v)_x \tag{80}$$

and

$$\begin{aligned}\hat T^{\delta_*}(u,u_x,v_x) &= \mathcal{O}((|u|+|u_x|)|v_x|),\\ \hat T^{\delta_*}(u,u_x,v_x)_x &= \mathcal{O}((|u_x|+|u_{xx}|)|v_x| + (|u_x|+|u_{xx}|)|v_{xx}|)\end{aligned} \tag{81}$$

for $|u|$ sufficiently small.

Observing that $\hat L(u)$ may be expanded as a nondivergence-form operator with $C^{0+\alpha}$ coefficients, we may obtain by standard Schauder or parametrix theory, similarly as in Lemma 1, both short-time existence for (79) and also the smoothing estimates

$$\begin{aligned}|v|_{C^2}(t) &\le Ct^{-1+\alpha}(|u_0|_{C^{0+\alpha}} + |u|_\alpha),\\ \sup_z |v_x|(\theta+\psi_1+\psi_2)^{-1}(z,t) &\le Ct^{-1/2}\\ &\quad\times\Big(|u|_{\zeta_1} + \sup_z|u_0|(\theta+\psi_1+\psi_2)^{-1}(z,0)\Big)\end{aligned} \tag{82}$$

for $0 \le t \le 1$ and

$$|v|_{C^2}(t) \le C(|u_0|_{C^{0+\alpha}} + |u|_\alpha + |v|_{C^1}(t-1)),$$
$$\sup_z |v_x|(\theta + \psi_1 + \psi_2)^{-1}(z,t) \le C \times \Big(|u|_{\zeta_1} + \sup_z |v|(\theta + \psi_1 + \psi_2)^{-1}(z, t-1)\Big) \tag{83}$$

for $t \ge 1$. Substituting these bounds in (78), and performing estimates similarly as in Section 5, we find by the same type of continuous induction scheme that $\mathcal{T}$ is well-defined, and bounded from a sufficiently large ball about the origin to itself in $(\mathcal{B}_1 \cap \mathcal{C}) \times \mathbb{R}^1$. Moreover, similar estimates yield that $\mathcal{T}$ restricted to this same ball in $(\mathcal{B}_1 \cap \mathcal{C}) \times \mathbb{R}^1$ intersected with a sufficiently small ball in $\mathcal{B}_\infty \times \mathbb{R}$ is contractive in the rescaled ζ_1 norm

$$|(i,j,k,l)|_{**} := |(i,j,k)|_{\zeta_1} + C|l|, \tag{84}$$

with $|\mathcal{T}(0)|_{\mathcal{B}_\infty \times \mathbb{R}} \le CE_0$. Combining these facts, we obtain a unique fixed-point solution in $(\mathcal{B}_1 \cap \mathcal{C}) \times \mathbb{R}^1$, for which

$$|(u, \delta, \dot\delta)|_\zeta \le CE_0,$$

yielding similarly as in Section 5 a global solution of the perturbation equations satisfying the claimed bounds. We omit the details. ∎

7 Technical Lemmas

We complete our analysis, finally, by the proof of the deferred technical lemmas used in Sections 3–6.

7.1 Short-time theory

Proof of Lemma 1. We carry out the proof for equations (22)–(23). The proof for equations (50), (41) goes similarly. By standard Schauder or parametrix theory [15, 13], there exists a unique solution $\tilde u \in C^{2+\alpha}(x) \cap C^{1+\frac{\alpha}{2}}(t)$ for $t > 0$ sufficiently small of the original (unshifted) equation (2), extending so long as $|\tilde u|_{C^{0+\alpha}(x)}$ remains bounded and satisfiying uniform bounds

$$|\tilde u|_{C^{0+\alpha}(x)} \le C, \quad |\tilde u|_{C^1(x)} \le C\Big(\frac{t}{1+t}\Big)^{-\frac12+\alpha}, \quad |\tilde u|_{C^2(x)} \le C\Big(\frac{t}{1+t}\Big)^{-1+\alpha} \tag{85}$$

depending only $\sup |\tilde u|_{C^{0+\alpha}(x)}$.

This in turn determines $\dot\delta$, hence δ, through (24) by a straightforward contraction-mapping/continuation argument. (Note: in establishing contractivity for small time of the righthand side of (24), we must establish bounds of form

$$C\int_0^t \int_{-\infty}^{+\infty} |e_y|(|\tilde u_x|^2 + |\tilde u||\tilde u_{xx}|)\,dy\,ds \le \tilde C \int_0^t (|\tilde u_x|^2_{L^\infty} + |\tilde u|_{L^\infty}|\tilde u_{xx}|_{L^\infty})\,ds < 1$$

for $t > 0$ sufficiently small, which follow by $|e_y|_{L^1} \le C$, a consequence of Remark 6, and integrability of $|\tilde u_x|^2_{L^\infty} + |\tilde u|_{L^\infty}|\tilde u_{xx}|_{L^\infty}$, in turn a consequence of estimates (85).) From (24), moreover, we easily obtain that $\delta \in C^{1+\frac{\alpha}{2}}$, and uniformly bounded for short time, so that perturbation $\tilde u(x+\delta(t), t) - \bar u(x)$, or equivalently its shift $\tilde u(x,t) - \bar u(x - \delta(t))$, enjoys the same regularity properties as $\tilde u$. (Recall that $\bar u \in C^3$, as a solution of the C^2 traveling-wave ODE.) Moreover, $|\tilde u|_{C^{0+\alpha}}$ remains uniformly bounded so long as $|u|_{C^{0+\alpha}}$ does, and vice versa, since their difference $\bar u(x - \delta(t))$ is uniformly bounded in $C^{0+\alpha}$.

This verifies the first assertion. To verify the second, observe for fixed t_0, τ that $u(x,t)$ for $t_0 - \tau \le t \le t_0$ by Duhamel's principle satisfies

$$u(x,t) = \int_{-\infty}^{\infty} \tilde G(x, t - (t_0 - \tau); y) u(y, t_0 - \tau)\,dy + \int_{t_0-\tau}^t \int_{-\infty}^{\infty} \tilde G_y(x, t-s; y)(Q(u, u_x) + \dot\delta u)(y,s)\,dy\,ds$$

and therefore

$$u_x(x,t) = \int_{-\infty}^{\infty} \tilde{G}_x(x, t-(t_0-\tau); y) u(y, t_0-\tau)\, dy + \int_{t_0-\tau}^{t} \int_{-\infty}^{\infty} \tilde{G}_x(x, t-s; y)(Q(u,u_x)_x + \dot{\delta} u_x)(y,s)\, dy\, ds.$$

Combining (21) and (74), we obtain

$$\begin{aligned} |u_x(x,t)| &\le \int_{-\infty}^{\infty} |\tilde{G}_x(x, t-(t_0-\tau); y)||u(y, t_0-\tau)|\, dy \\ &+ C \int_{t_0-\tau}^{t} \int_{-\infty}^{\infty} |\tilde{G}_x(x, t-s; y)| \\ &\times \Big(|u_x|(s-(t_0-\tau))^{-\frac{1}{2}+\alpha} + |u|(s-(t_0-\tau))^{-1+\alpha} \Big)(y,s)\, dy\, ds, \end{aligned} \tag{86}$$

from which we readily obtain

$$|u_x \Psi^{-1}(x,t)| \le (t-(t_0-\tau))^{-\frac{1}{2}} \sup_z |u(z, t_0-\tau)\Psi^{-1}(z, t_0-\tau)|$$

for $t_0 - \tau \le t \le t_0$ by a contraction argument based on the convolution estimates of Lemmas 5–7, thus verifying (25). (By uniqueness, the fixed point obtained by contraction mapping must in fact be u.) See Lemma 5.1 [52] or Lemma 11.5, [53] for similar arguments.

Likewise, we obtain by quite similar argument the Hölder bound

$$|\partial_t^{\frac{\alpha}{2}} u||\Psi^{-1}(x,t)| \le (t-(t_0-\tau))^{-\alpha} \sup_z |u(z, t_0-\tau)\Psi^{-1}(z, t_0-\tau)|,$$

where $\partial_t^{\frac{\alpha}{2}} u(x,t)$ denotes $\limsup_{\epsilon\to 0} |u(x,\cdot)|_{C^{0+\frac{\alpha}{2}}[t-\epsilon, t+\epsilon]}$ with some abuse of notation. Combining this fact with the uniform bound $|\Psi_t \Psi^{-1}| \le C$ obtainable by direct calculation, we obtain the claimed continuity of $\sup_z |u\Psi^{-1}(z,\cdot)|$. Indeed, we obtain Hölder continuity, $C^{0+\frac{\alpha}{2}}$. ∎

7.2 Integral estimates

Throughout the analysis, we will make use of the following lemmas.

Lemma 5. *Let $f(y) \ge 0$ be a bounded, nonincreasing function on $\mathbb{R}_+$, and also let $f \in L^1(\mathbb{R})$. Then for any $a > 0$ and $z > 0$, and for any $\omega > 1$,*

$$\int_0^{+\infty} a^{1/2} e^{-a(z-y)^2} f(y) dy \le (\frac{\sqrt{\pi}}{2} f(z/\omega)) \wedge (a^{1/2}\|f\|_{L^1(\mathbb{R})}) + [(\frac{\sqrt{\pi}}{2}\|f\|_{L^\infty(\mathbb{R})}) \wedge (a^{1/2}\|f\|_{L^1(\mathbb{R})})] e^{-a\gamma z^2},$$

for any $\gamma < (1-\frac{1}{\omega})^2$, and where $\wedge$ represents minimum.

Proof of Lemma 5. Lemma 5 is proven as Lemma 6.3 in [23]. ∎

The following two lemmas, useful in analyzing the $\theta(x,t)$ terms in the nonlinearity Ψ, can be proven in straightforward fashion by completing an appropriate square.

Lemma 6. *For any x, y, s, t, M_1, M_2 a, and b, we have*

$$\frac{(x-y-a(t-s))^2}{M_1(t-s)} + \frac{(y-bs)^2}{M_2 s} = \frac{(x-a(t-s)-bs)^2}{M_1(t-s)+M_2 s} + \frac{M_1(t-s)+M_2 s}{M_1 M_2 s(t-s)}\Big(y - \frac{(xM_2 s - (aM_2+bM_1)(t-s)s)}{M_1(t-s)+M_2 s}\Big)^2.$$

Lemma 7. *For any x, y, s, t, M_1, M_2 a, b, and c we have*

$$\begin{aligned}\frac{(x-\frac{a}{b}y-a(t-s))^2}{M_1(t-s)}+\frac{(y-cs)^2}{M_2 s}&=\frac{(x-a(t-s)-c\frac{a}{b}s)^2}{M_1(t-s)+M_2(\frac{a}{b})^2 s}\\&+\frac{M_1(t-s)+M_2(\frac{a}{b})^2 s}{M_1 M_2(\frac{a}{b})^2 s(t-s)}\\&\times\Big(\frac{a}{b}y-\frac{(a(\frac{a}{b})^2M_2+(a(t-s)+c\frac{a}{b}s)M_1)s(t-s)-x(\frac{a}{b})^2M_2 s}{M_1(t-s)+(\frac{a}{b})^2M_2 s}\Big)^2.\end{aligned}$$

Proof of Lemma 2. In each case of Lemma 2, we proceed in the case $x, y \le 0$. The case $y \le 0 \le x$ is similar to the reflection estimate for $x, y \le 0$. The case $y > 0$ is entirely symmetric. (See Remark 5 for a discussion of our terminology regarding convection, reflection, and transmission kernels. We will designate the case $\gamma = 0$ as the Lax case and the case $\gamma = 1$ as the undercompressive case.)

For the first estimate in Lemma 2, we consider the integrals

$$\int_{-\infty}^0 |\tilde{G}(x,t;y)|(1+|y|)^{-3/2}dy.$$

Convection estimate. For the convection kernel, according to Lemma 5, we have

$$\begin{aligned}&\int_{-\infty}^0 t^{-1/2}e^{-\frac{(x-y-a_k^- t)^2}{Mt}}(1+|y|)^{-3/2}dy\\&\le C\Big(t^{-1/2}\wedge(1+|x-a_k^- t|)^{-3/2}+(1+t)^{-1/2}e^{-\frac{(x-a_k^- t)^2}{M' t}}\Big),\end{aligned}$$

where $M' > M$ can be taken as close to M as we choose (by choosing the γ of Lemma 5 sufficiently close to 1). In the event that x and a_k^- have opposite signs, we have decay of the form $(|x| + t)^{-3/2}$, which can be absorbed by the claimed estimates. In the event that x and a_k^- have the same sign, the terms $(1+t)^{-1/2}e^{-\frac{(x-a_k^- t)^2}{Mt}}$ are exactly the $\theta(x,t)$. In order to see that the expression $t^{-1/2}\wedge(1+|x-a_k^- t|)^{-3/2}$ can be absorbed into the sum $\theta+\psi_1+\psi_2$, we first observe that for $|x-a_k^- t|\le C\sqrt{t}$,

$$t^{-1/2}\wedge(1+|x-a_k^- t|)^{-3/2}\le C_1(1+t)^{-1/2}e^{-\frac{(x-a_k^- t)^2}{Mt}},$$

for some constant C_1. On the other hand, for $|x-a_k^- t| > C\sqrt{t}$,

$$(1+|x-a_k^- t|)^{-3/2}\le t^{-1/2}(1+|x-a_k^- t|)^{-1/2},$$

which is sufficient in the case $|x| \le |a_1^-|t$. Finally, for $x\in[\frac{a_k^-}{2}t, 2a_k^- t]$, $t^{-1/2}\le C(|x|+t)^{-1/2}$, while for $x\notin[\frac{a_k^-}{2}t, 2a_k^- t]$, there can be only limited cancellation between x and $a_k^- t$, and we have

$$(1+|x-a_k^- t|)^{-3/2}\le C(1+|x|+t)^{-3/2}.$$

For $|x|\ge|a_1^-|t$, we analyze the case $|x-a_1^- t|\le C\sqrt{t}$ as above, leaving the case $x\le a_1^- t - C\sqrt{t}$, for which

$$t^{-1/2}\wedge(1+|x-a_1^- t|)^{-3/2}\le C(1+|x-a_1^- t|+t^{1/2})^{-3/2}.$$

Reflection estimate. In the case of the reflection kernel, Lemma 5 provides the estimate

$$\begin{aligned}&\int_{-\infty}^0 t^{-1/2}e^{-\frac{(x-\frac{a_j^-}{a_k^-}y-a_j^- t)^2}{Mt}}(1+|y|)^{-3/2}dy\\&\le C\Big(t^{-1/2}\wedge(1+|x-a_j^- t|)^{-3/2}+(1+t)^{-1/2}e^{-\frac{(x-a_j^- t)^2}{M' t}}\Big),\end{aligned}$$

which can be analyzed exactly as above. For the transmission kernel (and $x < 0$), we have exponential decay in x. In the event that $|x| \geq \epsilon t$ for some $\epsilon > 0$, we have exponential decay in both t and x, which can be subsumed. In the event that $|x| \leq \epsilon t$, we have

$$\begin{aligned}\int_{-\infty}^{0} & t^{-1/2} e^{-\frac{(x-\frac{a_j^+}{a_k^-}y - a_j^+ t)^2}{Mt}} e^{-\eta|x|}(1+|y|)^{-3/2} dy \\ & \leq C(1+t)^{-3/2} e^{-\eta|x|},\end{aligned}$$

which can be subsumed into the claimed estimates. The cases for $x \geq 0$ are similar.

For the second estimate of Lemma 2, we consider integrals of the form

$$\int_{-\infty}^{0} |e_t(y,t)|(1+|y|)^{-3/2} dy,$$

where according to Remark 6

$$|e_t(y,t)| \leq Ct^{-1/2} \sum_{a_k^- > 0} e^{-\frac{(y+a_k^- t)^2}{Mt}}.$$

According to Lemma 5, we can estimate this integral as,

$$\begin{aligned}\int_{-\infty}^{0} |e_t(y,t)|(1+|y|)^{-3/2} dy & \leq C \int_{-\infty}^{0} t^{-1/2} e^{-\frac{(y+a_k^- t)^2}{Mt}} (1+|y|)^{-3/2} dy \\ & \leq C(1+t)^{-3/2}.\end{aligned}$$

The third estimate of Lemma 2 follows directly from the boundedness of $e(y,t)$.

For the final estimate of Lemma 2, we consider integrals of the form

$$\int_{-\infty}^{0} |e(y,t) - e(y,+\infty)|(1+|y|)^{-3/2},$$

where according to Remark 6,

$$|e(y,t) - e(y,+\infty)| \leq C \text{errfn}\left(\frac{|y| - at}{M\sqrt{t}}\right)$$

for some C, M, $a > 0$. Computing directly, we have, then

$$\begin{aligned}\int_{-\infty}^{0} & |e(y,t) - e(y,+\infty)|(1+|y|)^{-3/2}\, dy \\ & \leq C \int_{-\infty}^{+\infty} \text{errfn}\left(\frac{|y| - at}{M\sqrt{t}}\right)(1+|y|)^{-3/2}\, dy \\ & = \frac{1}{2\pi} C \int_{-\infty}^{0} \Big(\int_{-\infty}^{\frac{-y-at}{M\sqrt{t}}} e^{-z^2} dz\Big)(1+|y|)^{-3/2} dy \\ & = \frac{1}{2\pi} C \int_{-\infty}^{-\frac{a}{2}t} \Big(\int_{-\infty}^{\frac{-y-at}{M\sqrt{t}}} e^{-z^2} dz\Big)(1+|y|)^{-3/2} dy \\ & + \frac{1}{2\pi} C \int_{-\frac{a}{2}t}^{0} \Big(\int_{-\infty}^{\frac{-y-at}{M\sqrt{t}}} e^{-z^2} dz\Big)(1+|y|)^{-3/2} dy \\ & \leq C_1(1+t)^{-1/2} + C_2 e^{-\frac{a^2}{8M^2}t}.\end{aligned}$$

This completes the proof for $x, y \leq 0$. As discussed above, the cases in which either x or y is positive follow similarly. ∎

Proof of Lemma 3. For Lemma 3, the proof of each estimate requires the analysis of several cases. We proceed by carrying out detailed calculations in the most delicate cases and sufficing to indicate the appropriate arguments in the others. In particular, we will always consider the case $x, y \le 0$. The case $y \le 0 \le x$ is similar (though certainly not identical) to the reflection case for $x, y \le 0$. The estimates for $y \ge 0$ are entirely symmetric. For the nonlinearity Ψ we observe the inequality

$$\Psi \le C(1+s)^{1/2} s^{-1/2} (\theta^2 + \psi_1^2 + \psi_2^2).$$

Nonlinearity θ^2. We begin by estimating convolutions of the form

$$\int_0^t \int_{-\infty}^0 |\tilde{G}_y(x, t-s; y)| (1+s)^{1/2} s^{-1/2} \theta(y,s)^2 dy ds.$$

Lax convection, $x \le 0$. For $a_j^- < 0$, we consider integrals of the form

$$\int_0^t \int_{-\infty}^0 (t-s)^{-1} e^{-\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}} (1+s)^{-1/2} s^{-1/2} e^{-\frac{(y-a_j^- s)^2}{Ms}} dy ds,$$

where the constant M arising in θ^2 is larger than the exact constant, $L/2$ (making the term an upper estimate). According to Lemma 6, we have

$$\begin{aligned}
&\int_0^t \int_{-\infty}^0 (t-s)^{-1} e^{-\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}} (1+s)^{-1/2} s^{-1/2} e^{-\frac{(y-a_j^- s)^2}{Ms}} dy ds \\
&= \int_0^t \int_{-\infty}^0 (t-s)^{-1} e^{-\frac{(x-a_k^-(t-s)-a_j^- s)^2}{Mt}} (1+s)^{-1/2} s^{-1/2} \\
&\quad \times e^{-\frac{t}{Ms(t-s)}\left(y - \frac{xs-(a_k^-+a_j^-)s(t-s)}{Mt}\right)^2} dy ds \\
&\le C t^{-1/2} \int_0^t (t-s)^{-1/2} (1+s)^{-1/2} e^{-\frac{(x-a_k^-(t-s)-a_j^- s)^2}{Mt}} ds.
\end{aligned}$$

The convection rate a_k^- can be either positive or negative, so we divide the analysis into three cases: (1) $a_j^- < 0 < a_k^-$, (2) $a_k^- \le a_j^- < 0$, and (3) $a_j^- < a_k^- < 0$. For the first, we observe that for $|x| \ge |a_j^-|t$, we can write

$$x - a_k^-(t-s) - a_j^- s = (x - a_j^- t) - (a_k^- - a_j^-)(t-s).$$

In this last expression, $x - a_j^- t$ and $-(a_k^- - a_j^-)(t-s)$ are both negative and cannot cancel, and we can compute

$$\begin{aligned}
&C t^{-1/2} \int_0^t (t-s)^{-1/2} (1+s)^{-1/2} e^{-\frac{(x-a_k^-(t-s)-a_j^- s)^2}{Mt}} ds \\
&\le C t^{-1/2} \int_0^t (t-s)^{-1/2} (1+s)^{-1/2} e^{-\frac{(x-a_j^- t)^2}{Mt}} ds \\
&\le C(1+t)^{-1/2} e^{-\frac{(x-a_j^- t)^2}{Mt}}.
\end{aligned}$$

(We observe that the seeming blow-up of $t^{-1/2}$ as $t \to 0$ is compensated for by the interval of integration $s \in [0,t]$.) In the event that $|x| \le |a_j^-|t$, we further divide the analysis into cases $s \in [0, t/2]$ and $s \in [t/2, t]$. For $s \in [0, t/2]$, we write

$$x - a_k^-(t-s) - a_j^- s = (x - a_k^- t) + (a_k^- - a_j^-)s,$$

for which $x - a_k^- t \le 0$ and $(a_k^- - a_j^-)s$ have opposite signs and cancellation occurs. In this way, we have the balance estimate

$$\begin{aligned}
&(1+s)^{-1/2} e^{-\frac{(x-a_k^-(t-s)-a_j^- s)^2}{Mt}} \\
&\le C\Big[(1+|x-a_k^- t|)^{-1/2} e^{-\frac{(x-a_k^-(t-s)-a_j^- s)^2}{Mt}} + (1+s)^{-1/2} e^{-\frac{(x-a_k^- t)^2}{M' t}}\Big].
\end{aligned}$$

We observe that this kind of balance estimate is contained within the proof of Lemma 5, but that the lemma cannot quite be directly applied, because we first must recognize which variables balance. In this way, the current analysis is a refinement of the analysis of [23]. We also observe that it is critical that for $s \in [0, t/2]$ the cancellation comes from s (which on this interval does not yield t decay), while for $s \in [t/2, t]$ it will be critical that the cancellation comes from $(t-s)$. Indeed, at a purely technical level, this observation drives the analysis. Computing directly, we now have

$$
\begin{aligned}
&t^{-1/2} \int_0^{t/2} (t-s)^{-1/2}(1+s)^{-1/2} e^{-\frac{(x-a_k^-(t-s)-a_j^- s)^2}{Mt}} ds \\
&\le C t^{-1/2} \int_0^{t/2} (t-s)^{-1/2} \Big[(1+|x-a_k^- t|)^{-1/2} e^{-\frac{(x-a_k^-(t-s)-a_j^- s)^2}{Mt}} ds \\
&\quad + (1+s)^{-1/2} e^{-\frac{(x-a_k^- t)^2}{M' t}} \Big] ds.
\end{aligned}
$$

Finally, observing that in this case $|x - a_k^- t| \ge c(|x| + t)$, and that $t \ge |x|/|a_j^-|$ we determine an estimate by

$$(1 + t + |x|)^{-1},$$

which can be subsumed into ψ_1. In the case $s \in [t/2, t]$ we argue similarly, beginning with the relation

$$x - a_k^-(t-s) - a_j^- s = (x - a_j^- t) - (a_k^- - a_j^-)(t-s).$$

For the second case ($a_j^- \le a_k^- < 0$), we again begin with the subcase $|x| \ge |a_j^-|t$, for which the argument for the case $a_j^- < 0 < a_k^-$ holds. Similarly, for the subcase $|x| \le |a_k^-|t$ we write

$$x - a_k^-(t-s) - a_j^- s = (x - a_k^- t) + (a_k^- - a_j^-)s,$$

for which again there is no cancellation, and we obtain an estimate by θ. In the event that $|a_k^-|t \le |x| \le |a_j^-|t$, we proceed in the cases $s \in [0, t/2]$ and $s \in [t/2, t]$ exactly as in the case $a_j^- < 0 < a_k^-$. The third case, $a_k^- < a_j^- < 0$ follows similarly, though here we begin with the cases $|x| \ge |a_k^-|t$ and $|x| \le |a_j^-|t$.

Lax reflection, $x \le 0$. For $a_l^- < 0$, $a_j^- < 0$, and $a_k^- > 0$, we compute from Lemma 7 (followed by direct integration)

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^0 (t-s)^{-1} e^{-\frac{(x - \frac{a_j^-}{a_k^-} y - a_j^-(t-s))^2}{M(t-s)}} (1+s)^{-1/2} s^{-1/2} e^{-\frac{(y - a_l^- s)^2}{Ms}} dyds \\
&\le C t^{-1/2} \int_0^t (t-s)^{-1/2} (1+s)^{-1/2} e^{-\frac{(x - a_j^-(t-s) - a_l^- \frac{a_j^-}{a_k^-} s)^2}{M((t-s) + (\frac{a_j^-}{a_k^-})^2 s)}} ds.
\end{aligned}
$$

In the case $|x| \ge |a_j^-|t$, we write

$$x - a_j^-(t-s) - a_l^- \frac{a_j^-}{a_k^-} s = (x - a_j^- t) + (a_j^- - a_l \frac{a_j^-}{a_k^-})s,$$

for which there is no cancellation and we obtain an estimate of the form of a diffusion wave. We observe that for $|a_j^-/a_k^-| > 1$ the estimate arising from this integral will in general be broader than the diffusion wave we began with (we will get a constant greater than L). We can proceed, however, by dividing the integration over s into cases, $s \in [0, t/C]$, C sufficiently large, for which we have control over the breadth of the diffusion wave, and $s \in [t/C, t]$ for which we have exponential decay in t, which can be subsumed by our estimates. (For $x \ge C_1 t$, C_1 sufficiently large, we clearly have exponential decay in both $|x|$ and t; hence, we can take decay in t to give similar decay in $|x|$).

For the case $|x| \le |a_j^-|t$, we first consider the case $s \in [0, t/C]$, for which we write

$$x - a_j^-(t-s) - a_l \frac{a_j^-}{a_k^-} s = (x - a_j^- t) + (a_j^- - a_l^- \frac{a_j^-}{a_k^-})s,$$

and proceed as in the Lax convection case with $s \in [0, t/2]$. For $s \in [t/C, t - t/C]$, we clearly have $(t + |x|)^{-1}$ decay, which can be subsumed, while for $s \in [t - t/C, t]$ we write

$$x - a_j^-(t-s) - a_l^- \frac{a_j^-}{a_k^-} s = (x - a_l^- \frac{a_j^-}{a_k^-} s) - a_j^-(t-s),$$

and proceed as in the Lax convection case with $s \in [t/2, t]$.

Lax transmission, $x \le 0$. For $a_l^- < 0$, $a_k^- > 0$, and $a_j^+ > 0$, we consider integrals of the form

$$\int_0^t \int_{-\infty}^0 (t-s)^{-1} e^{-\frac{(x - \frac{a_j^+}{a_k^-} y - a_j^+ t)^2}{Mt}} e^{-\eta|x|} (1+s)^{-1/2} s^{-1/2} e^{-\frac{(y - a_l^- s)^2}{Ms}} dyds$$

$$\le C t^{-1/2} e^{-\eta|x|} \int_0^t (t-s)^{-1/2} (1+s)^{-1/2} e^{-\frac{(x - a_j^+(t-s) - a_l^- \frac{a_j^+}{a_k^-} s)^2}{M((t-s) + (\frac{a_j^-}{a_k^-})^2) s}} ds.$$

In this case, for $|x| \ge \epsilon t$, and fixed $\epsilon > 0$, we have exponential decay in both $|x|$ and t, which can be subsumed into our estimates. In the case $|x| \le \epsilon t$, for the case $s \in [0, t/2]$, we write

$$x - a_j^+(t-s) - a_l^- \frac{a_j^+}{a_k^-} s = (x - a_j^+ t) + (a_j^+ - a_l^- \frac{a_j^+}{a_k^-}) s,$$

and proceed as in the Lax convection case with $s \in [0, t/2]$, while for $s \in [t/2, t]$ we write

$$x - a_j^+(t-s) - a_l^- \frac{a_j^+}{a_k^-} s = (x - a_l^- \frac{a_j^+}{a_k^-} t) + (a_l^- \frac{a_j^+}{a_k^-} - a_j^+)(t-s),$$

and proceed as in the Lax convection case with $s \in [t/2, t]$.

Undercompressive convection, $x \le 0$. For $a_j^- < 0$, we consider integrals of the form

$$\int_0^t \int_{-\infty}^0 (t-s)^{-1/2} e^{-\frac{(x - y - a_k^-(t-s))^2}{M(t-s)}} e^{-\eta|y|} (1+s)^{-1/2} s^{-1/2} e^{-\frac{(y - a_j^- s)^2}{Ms}} dyds,$$

for which we observe the inequality

$$e^{-\eta|y|} e^{-\frac{(y - a_j^- s)^2}{Ms}} \le C e^{-\eta_1 |y|} e^{-\eta_2 s}.$$

We can now employ Lemma 6 and proceed as in the Lax convection case, taking advantage of the integrability of $e^{-\eta_1|y|}$ in y and the integrability of $e^{-\eta_2 s}$ in s. Estimates in the undercompressive reflection case and undercompressive transmission case follow similarly. This concludes the analysis of the nonlinearity θ^2.

Nonlinearity ψ_1^2. We estimate convolutions of the form

$$\int_0^t \int_{-\infty}^0 |\tilde{G}_y(x, t-s; y)| (1+s)^{1/2} s^{-1/2} \psi_1^2 dyds.$$

Lax convection, $x < 0$. For $a_j^- < 0$, we consider integrals of the form

$$\int_0^t \int_{a_1^- s}^0 (t-s)^{-1} e^{-\frac{(x - y - a_k^-(t-s))^2}{M(t-s)}} (1+s)^{1/2} s^{-1/2}$$
$$\times (1 + |y| + s)^{-1} (1 + |y - a_j^- s|)^{-1} dyds.$$

Clearly, the primary new element here is that we no longer have the precision afforded by Lemmas 6 and 7. We proceed instead through balance estimates similar again to those that arise in the proof of Lemma 5.

We consider three subcases: (1) $a_j^- < 0 < a_k^-$, (2) $a_j^- \le a_k^- < 0$, and (3) $a_k^- < a_j^- < 0$, beginning with the first. We begin by considering the interval $|x| \ge |a_1^-|t$, on which we write

$$x - y - a_k^-(t-s) = (x - a_1^- t) - (y - a_1^- t) - a_k^-(t-s). \tag{87}$$

For $s \in [0,t]$ and $y \in [a_1^- s, 0]$, we have $y - a_1^- t \ge 0$ and consequently each term in (87) is negative, and we have no cancellation. Integrating in straightfoward fashion, then, we can conclude an estimate by $C\theta(x,t)$. For $|x| \le |a_1^-|t$, we begin by writing

$$x - y - a_k^-(t-s) = (x - a_k^-(t-s) - a_j^- s) - (y - a_j^- s),$$

from which we see that either $x - y - a_k^-(t-s)$ is near $x - a_k^-(t-s) - a_j^- s$ or

$$|y - a_j^- s| \ge \epsilon |x - a_k^-(t-s) - a_j^- s|,$$

for some $\epsilon > 0$. More precisely, we have the estimate

$$\begin{aligned}
&(1 + |y - a_j^- s|)^{-1/2} e^{-\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}} \\
&\quad \le C\Big[(1 + |x - a_k^-(t-s) - a_j^- s|)^{-1/2} e^{-\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}} \\
&\quad + (1 + |y - a_j^- s|)^{-1/2} e^{-\frac{(x-a_k^-(t-s)-a_j^- s)^2}{M'(t-s)}} e^{-\epsilon\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}}\Big].
\end{aligned} \tag{88}$$

For the second piece of this estimate, we have

$$\begin{aligned}
&\int_0^t \int_{a_1^- s}^0 (t-s)^{-1} e^{-\frac{(x-a_k^-(t-s)-a_j^- s)^2}{M'(t-s)}} e^{-\epsilon\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}} \\
&\quad \times (1+s)^{1/2} s^{-1/2} (1 + |y| + s)^{-1} (1 + |y - a_j^- s|)^{-1} dy ds.
\end{aligned}$$

On the interval $s \in [0, t/2]$, we integrate $(1 + |y - a_j^- s|)^{-1}$ in y, while on the interval $s \in [t/2, t]$ we integrate the remaining Gaussian kernel. We obtain an estimate, then, by

$$\begin{aligned}
&C\int_0^{t/2} t^{-1}(1+s)^{1/2} s^{-1/2} \log(1 + C's) e^{-\frac{(x-a_k^-(t-s)-a_j^- s)^2}{M'(t-s)}} ds \\
&\quad + C\int_{t/2}^t (1+t)^{-1}(t-s)^{-1/2} e^{-\frac{(x-a_k^-(t-s)-a_j^- s)^2}{M'(t-s)}} ds,
\end{aligned}$$

both of which can now be analyzed by the methods of the Lax convection case with nonlinearity θ^2.

The critical new estimate in this case is

$$\begin{aligned}
&\int_0^t \int_{a_1^- s}^0 (t-s)^{-1} e^{-\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}} (1 + |x - a_k^-(t-s) - a_j^- s|)^{-1/2} \\
&\quad \times (1+s)^{1/2} s^{-1/2} (1 + |y| + s)^{-1} (1 + |y - a_j^- s|)^{-1/2} dy ds.
\end{aligned}$$

For $|x| \ge |a_j^-|t$ (though $|x| \le |a_1^-|t$; we recall that the case $|x| \ge |a_1^-|t$ has already been considered), we write

$$x - a_k^-(t-s) - a_j^- s = (x - a_j^- t) - (a_k^- - a_j^-)(t-s),$$

for which we have no cancellation and straightfoward integration provides an estimate by $(1+t)^{-1/2}(1 + |x - a_k^- t|)^{-1/2}$. For $|x| \le |a_j^-|t$, we divide the analysis into the cases $s \in [0, t/2]$ and $s \in [t/2, t]$. In the case $s \in [0, t/2]$, we write

$$x - a_k^-(t-s) - a_j^- s = (x - a_k^- t) - (a_k^- - a_j^-)s,$$

for which (observing that for $x<0$ and $a_k^-<0$, $|x-a_k^-t|=|x|+|a_k^-|t$),

$$\begin{aligned}(1+|x-a_k^-(t-s)-a_j^-s|)^{-1/2}&(1+|y|+s)^{-1/2}\\ \leq C\Big[&(1+|x|+t)^{-1/2}(1+|y|+s)^{-1/2}\\ +(1+|x-a_k^-(t-s)-a_j^-s|)^{-1/2}&(1+|y|+|x|+t)^{-1/2}\Big].\end{aligned}\tag{89}$$

For the second estimate in (89), we integrate $(1+|y-a_j^-s|)^{-1/2}$ to find

$$\begin{aligned}&\int_0^{t/2}\int_{a_1^-s}^0(t-s)^{-1}e^{-\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}}(1+s)^{1/2}s^{-1/2}(1+|y|+s)^{-1/2}\\ &\times(1+|y-a_j^-s|)^{-1/2}(1+|x-a_k^-(t-s)-a_j^-s|)^{-1/2}(1+|x|+t)^{-1/2}dyds\\ &\quad\leq Ct^{-1}(1+|x|+t)^{-1/2}\int_0^{t/2}(1+s)^{1/2}ds\\ &\quad\leq Ct^{-1/2}(1+|x|+t)^{-1/2}.\end{aligned}$$

The first estimate in (89) can be analyzed similarly. In the case $s\in[t/2,t]$, we have

$$x-a_k^-(t-s)-a_j^-s=(x-a_j^-t)-(a_k^--a_j^-)(t-s),$$

for which we have

$$\begin{aligned}(t-s)^{-1/2}&(1+|x-a_k^-(t-s)-a_j^-s|)^{-1/2}\\ &\leq C\Big[|x-a_j^-t|^{-1/2}(1+|x-a_k^-(t-s)a_j^-s|)^{-1/2}\\ &\quad+(t-s)^{-1/2}(1+|x-a_j^-t|)^{-1/2}\Big]\end{aligned}$$

For the first of these, we estimate

$$\begin{aligned}&\int_{t/2}^t\int_{a_j^-s}^0(t-s)^{-1/2}|x-a_j^-t|^{-1/2}(1+|x-a_k^-(t-s)a_j^-s|)^{-1/2}\\ &\times e^{-\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}}(1+s)^{1/2}s^{-1/2}(1+|y|+s)^{-1}dyds\\ &\quad\leq C(1+t)^{-1}|x-a_j^-t|^{-1/2}\\ &\quad\times\int_{t/2}^t(1+|x-a_k^-(t-s)a_j^-s|)^{-1/2}(1+s)^{1/2}s^{-1/2}ds\\ &\quad\leq C(1+t)^{-1/2}|x-a_j^-t|^{-1/2},\end{aligned}$$

and similarly for the second. We remark that the apparent blow-up at $x=a_j^-t$ is an artifact of the approach and can be removed by the observation that for $|x-a_j^-t|\leq C\sqrt{t}$, we can proceed by alternative estimates to get decay of form $\theta(x,t)$.

The remaining cases $a_j^-\leq a_k^-<0$ and $a_k^-<a_j^-<0$ follow similarly as in the Lax convection case for θ^2, with the arguments augmented by balance estimates along the lines of (88).

Lax reflection. For $a_l^-<0$, $a_k^->0$, and $a_j^-<0$, we consider the convolutions

$$\begin{aligned}&\int_0^t\int_{a_1^-s}^0(t-s)^{-1}e^{-\frac{(x-\frac{a_j^-}{a_k^-}y-a_j^-(t-s))^2}{M(t-s)}}\\ &\times(1+s)^{1/2}s^{-1/2}(1+|y|+s)^{-1}(1+|y-a_l^-s|)^{-1}dyds.\end{aligned}$$

For $|x|\geq|a_j^-|t$, we write

$$x-\frac{a_j^-}{a_k^-}y-a_j^-(t-s)=(x-a_j^-t)-(\frac{a_j^-}{a_k^-}y-a_j^-s).$$

Here, $x - a_j^- t < 0$ and $\frac{a_j^-}{a_k^-} y - a_j^- s > 0$, and hence we do not have cancellation, and we get an estimate by $C\theta(x,t)$. For $|x| \le |a_j^-|t$, we write

$$x - \frac{a_j^-}{a_k^-} y - a_j^-(t-s) = (x - a_j^-(t-s) - a_l^- \frac{a_j^-}{a_k^-} s) - \frac{a_j^-}{a_k^-}(y - a_l^- s),$$

from which we have the inequality

$$\begin{aligned}
&(1+|y-a_l^- s|)^{-1/2} e^{-\frac{(x - \frac{a_j^-}{a_k^-} y - a_j^-(t-s))^2}{M(t-s)}} \\
&\le C\Big[(1+|x - a_j^-(t-s) - a_l^- \frac{a_j^-}{a_k^-} s|)^{-1/2} e^{-\frac{(x - \frac{a_j^-}{a_k^-} y - a_j^-(t-s))^2}{M(t-s)}} \\
&+ (1+|y-a_l^- s|)^{-1/2} e^{-\frac{(x - a_j^-(t-s) - a_l^- \frac{a_j^-}{a_k^-} s)^2}{M'(t-s)}} e^{-\epsilon \frac{(x - \frac{a_j^-}{a_k^-} y - a_j^-(t-s))^2}{M(t-s)}}\Big].
\end{aligned}$$

We proceed now as in the Lax convection case for ψ_1^2, writing for $s \in [0, t/2]$

$$x - a_j^-(t-s) - a_l^- \frac{a_j^-}{a_k^-} s = (x - a_j^- t) + (a_j^- - a_l^- \frac{a_j^-}{a_k^-}) s$$

and for $s \in [t/2, t]$

$$x - a_j^-(t-s) - a_l^- \frac{a_j^-}{a_k^-} s = (x - a_l^- \frac{a_j^-}{a_k^-} t) + (a_l^- \frac{a_j^-}{a_k^-} - a_j^-)(t-s).$$

Lax transmission. For $a_l^- < 0$, $a_k^- > 0$, and $a_j^+ > 0$, we consider the convolutions

$$\begin{aligned}
&\int_0^t \int_{a_1^- s}^0 (t-s)^{-1} e^{-\frac{(x - \frac{a_j^+}{a_k^-} y - a_j^+(t-s))^2}{M(t-s)}} e^{-\eta|x|} \\
&\times (1+s)^{1/2} s^{-1/2} (1+|y|+s)^{-1} (1+|y-a_l^- s|)^{-1} dy ds.
\end{aligned}$$

In the case $|x| \ge \epsilon t$, some fixed $\epsilon > 0$, we have exponential decay in both $|x|$ and t, which can be subsumed. In the case $|x| \le \epsilon t$, we proceed almost exactly as in the Lax reflection case for ψ_1^2.

Undercompressive convection. For $a_j^- < 0$, we consider the convolutions

$$\begin{aligned}
&\int_0^t \int_{a_1^- s}^0 (t-s)^{-1/2} e^{-\frac{(x - y - a_k^-(t-s))^2}{M(t-s)}} e^{-\eta|y|} \\
&\times (1+s)^{1/2} s^{-1/2} (1+|y|+s)^{-1} (1+|y-a_l^- s|)^{-1} dy ds.
\end{aligned}$$

We observe here the inequality

$$e^{-\eta|y|}(1+|y-a_j^- s|)^{-1} \le C\Big[e^{-\eta_1|y|} e^{-\eta_2 s} + e^{-\eta|y|}(1+s)^{-1}\Big].$$

Integrating $e^{-\eta|y|}$ (or $e^{-\eta_1|y|}$), we now proceed similarly as in the analysis of the Lax convection case for ψ_1^2. Similarly, the undercompressive reflection and transmission estimates follow as in the Lax reflection and transmission esimates. This concludes the analysis for the nonlinearity ψ_1^2.

Nonlinearity ψ_2^2. We estimate convolutions of the form

$$\int_0^t \int_{-\infty}^{a_1^- s} |\tilde{G}_y(x, t-s; y)| (1+s)^{1/2} s^{-1/2} \psi_2(y,s)^2 dy ds.$$

Lax convection. For $a_1^- < 0$, we consider convolutions of the form

$$\int_0^t \int_{-\infty}^{a_1^- s} (t-s)^{-1} e^{-\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}} \times (1+s)^{1/2} s^{-1/2} (1+|y-a_1^- s| + s^{1/2})^{-3} dyds.$$

We write

$$x - y - a_k^-(t-s) = (x - a_k^-(t-s) - a_1^- s) - (y - a_1^- s),$$

for which

$$\begin{aligned}
&(1+|y-a_1^- s| + s^{1/2})^{-3/2} e^{-\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}} \\
&\leq C\Big[(1+|x-a_k^-(t-s)-a_1^- s| + s^{1/2})^{-3/2} e^{-\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}} \\
&+ (1+|y-a_1^- s| + s^{1/2})^{-3/2} e^{-\frac{(x-a_k^-(t-s)-a_1^- s)^2}{M'(t-s)}} e^{-\epsilon\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}}\Big].
\end{aligned}$$

For $|x| \geq |a_1^-|t$, we write

$$x - a_k^-(t-s) - a_j^- s = (x - a_1^- t) - (a_k^- - a_1^-)(t-s),$$

for which there is no cancellation and integration gives an estimate by

$$C_1(1+t)^{-1}\log(1+t)e^{-\frac{(x-a_1^- t)^2}{M't}} + C_2(1+t)^{-1/4}(1+|x-a_1^- t| + t^{1/2})^{-3/2}.$$

For $|x| \leq |a_1^-|t$, we consider only the case $a_k^- < 0$. The case $a_k^- > 0$ is similar. We first consider the additional subcase $|x| \leq |a_k^-|t$, for which we write

$$x - a_k^-(t-s) - a_1^- s = (x - a_k^- t) + (a_k^- - a_1^-)s.$$

Observing that there is no cancellation between these terms, we can proceed exactly as in the case $|x| \geq |a_1^-|t$ to get an estimate by $C(\theta(x,t) + \psi_2)$. For the case $|a_k^-|t \leq |x| \leq |a_j^-|t$, we divide the analysis into subcases $s \in [0, t/2]$ and $s \in [t/2, t]$. For $s \in [0, t/2]$, we write

$$x - a_k^-(t-s) - a_1^- s = (x - a_k^- t) + (a_k^- - a_1^-)s,$$

for which we have

$$\begin{aligned}
&(1+|x-a_k^-(t-s)-a_1^- s| + s^{1/2})^{-3/2} \\
&\leq C\Big[(1+|x-a_k^- t| + s^{1/2})^{-3/2} \\
&+ (1+|x-a_k^-(t-s)-a_1^- s| + |x-a_k^- t|^{1/2} + s^{1/2})^{-3/2}\Big].
\end{aligned} \tag{90}$$

For the first of the estimates, integrating $(1+|y-a_1^- s| + s^{1/2})^{-3/2}$, we estimate

$$\begin{aligned}
&\int_0^{t/2} \int_{-\infty}^{a_1^- s} (t-s)^{-1}(1+|x-a_k^-(t-s)-a_1^- s| + s^{1/2})^{-3/2} e^{-\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}} \\
&\times (1+s)^{1/2} s^{-1/2}(1+|y-a_1^- s| + s^{1/2})^{-3/2} dyds \\
&\leq Ct^{-1}\int_0^{t/2}(1+|x-a_k^- t| + s^{1/2})^{-3/2}(1+s)^{1/2}s^{-1/2}(1+s^{1/2})^{-1/2}ds \\
&\leq C(1+t)^{-3/4}(1+|x-a_k^- t|)^{-1/2}.
\end{aligned}$$

For $s \in [t/2, t]$, we write

$$x - a_k^-(t-s) - a_1^- s = (x - a_1^- t) - (a_k^- - a_1^-)(t-s),$$

for which we have

$$\begin{aligned}(t-s)^{-1/2}&\Big(1+|x-a_k^-(t-s)-a_1^-s|+s^{1/2}\Big)^{-3/2}\\ &\le C\Big[|x-a_1^-t|^{-1/2}\Big(1+|x-a_k^-(t-s)-a_1^-s|+s^{1/2}\Big)^{-3/2}\\ &+(t-s)^{-1/2}\Big(1+|x-a_1^-t|+s^{1/2}\Big)^{-3/2}\Big].\end{aligned}$$

For the first, integrating over the Gaussian kernel, we estimate

$$\begin{aligned}&\int_{t/2}^{t}\int_{-\infty}^{a_1^-s}(t-s)^{-1}(1+|x-a_k^-(t-s)-a_1^-s|+s^{1/2})^{-3/2}e^{-\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}}\\ &\times(1+s)^{1/2}s^{-1/2}(1+|y-a_1^-s|+s^{1/2})^{-3/2}dyds\\ &\le C(1+t^{1/2})^{-3/2}|x-a_1^-t|^{-1/2}\\ &\times\int_{t/2}^{t}(1+|x-a_k^-(t-s)-a_1^-s|+t^{1/2})^{-3/2}(1+s)^{1/2}s^{-1/2}ds\\ &\le C(1+t)^{-1/2}(1+|x-a_k^-t|)^{-1/2}.\end{aligned}$$

The second estimate in (90) can be analyzed similarly.

The Lax reflection and Lax transmission estimates follow similarly. Finally, for the undercompressive estimates in the case of nonlinearity ψ_2^2, we observe that for $y\in(-\infty,a_1^-s]$ exponential $|y|$ decay yields exponential s decay, and the estimates follow in straightforward fashion. This concludes the analysis for the nonlinearity ψ_2^2 and consequently for the nonlinearity

$$(1+s)^{1/2}s^{-1/2}(\theta+\psi_1+\psi_2)^2.$$

Nonlinearity $(1+s)^{-1}(\theta+\psi_1+\psi_2)$. We consider convolutions of the form

$$\int_0^t\int_{-\infty}^0|\tilde{G}_y(x,t-s;y)|(1+s)^{-1}(\theta+\psi_1+\psi_2)dyds.$$

Lax convection. For $a_j^-<0$, we consider integrals of the form

$$\int_0^t\int_{-\infty}^0(t-s)^{-1}e^{-\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}}(1+s)^{-1}s^{-1/2}e^{-\frac{(y-a_j^-s)^2}{Ls}}dyds.$$

We observe that this integral is better than the one analyzed in the Lax convection case with nonlinearity θ^2, except that the constant L in the diffusion kernel must now be kept. (In general, our balance estimates increase the size of this constant, see especially Lemma 5.) According to Lemma 6, we can estimate this integral by

$$Ct^{-1/2}\int_0^t(t-s)^{-1/2}(1+s)^{-1}e^{-\frac{1}{M(t-s)+Ls}(x-a_k^-(t-s)-a_j^-s)^2}ds.$$

For $|x|\ge|a_j^-|t$ and the case $a_j^-\le a_k^-$, we write

$$x-a_k^-(t-s)-a_j^-s=(x-a_j^-t)-(a_k^--a_j^-)(t-s),$$

for which there is no cancellation and integration yields an estimate by

$$(1+t)^{-1/2}e^{-\frac{(x-a_j^-t)^2}{M't}},$$

where $M<M'<L$. For $|x|\le|a_j^-|t$, we divide the analysis into the subcases $s\in[0,t/C_1]$ and $s\in[t/C_1,t]$ for some constant C_1 sufficiently large. In the event that $s\in[t/C_1,t]$, we integrate $(t-s)^{-1/2}$ to obtain the estimate

$$C(1+t)^{-1},$$

which can be subsumed. On the other hand if $s \in [0, t/C_1]$, we observe that by choice of C_1 the divisor $M(t-s) + Ls$ can be kept as close to Mt as we like. Since $M < L$ we can use this observation to recover the expected estimate. Otherwise, the analysis proceeds exactly as in the Lax convection case for nonlinearity θ^2.

The Lax reflection and Lax transmission estimates for nonlinearity $(1+t)^{-1}\theta$ follow from the Lax convection argument and Lemma 7. The undercompressive estimates follow similarly.

In the case of the nonlinearity $(1+s)^{-1}\psi_1$, we observe that for $y \in [a_1^- s, 0]$,

$$(1+s)^{-1} \le C(1+|y|+s)^{-1},$$

and so

$$(1+s)^{-1}\psi_1 \le C\psi_1^2.$$

Hence the convolution estimates for his nonlinearity follow from those for the nonlinearity ψ_1^2. Finally, estimates for the nonlinearity $(1+s)^{-1}\psi_2$ are straightfoward. This concludes the analysis of the first estimates of Lemma 3.

Excited term estimates. The remaining estimates of Lemma 3 regard the *excited* terms $e(y,t)$. For the integral

$$\int_0^t \int_{-\infty}^0 |e_{yt}(y, t-s)| \Psi(y,s) dy ds,$$

we have according to Remark 6

$$|e_{yt}(y,t)| \le C(t^{-1} + \gamma t^{-1/2} e^{-\eta|y|}) \sum_{a_k^- > 0} e^{-\frac{(y+a_k^- t)^2}{Mt}}.$$

We observe that these estimates correspond precisely with the Lax and undercompressive convection kernels with $x = 0$. In this way, we immediately obtain an estimate by

$$C(\theta + \psi_1 + \psi_2)(0,t) \le C_1(1+t)^{-1},$$

for some constant C_1. For the integral

$$\int_0^\infty \int_{-\infty}^0 |e_y(y, +\infty)| \Psi(y,s) dy ds,$$

we have from Remark 6

$$|e_y(y,+\infty)| \le C\gamma e^{-\eta|y|},$$

where as usual γ is 1 for undercompressive profiles and 0 otherwise. For the nonlinearity $(1+s)^{1/2}s^{-1/2}\theta^2$, we observe the inequality

$$e^{-\eta|y|} e^{-\frac{(y-a_j^- s)^2}{Ms}} \le C e^{-\eta_1|y|} e^{-\eta_2 s},$$

for some fixed $\eta_1 > 0$ and $\eta_2 > 0$. We estimate

$$\int_0^\infty \int_{-\infty}^0 e^{-\eta_1|y|} e^{-\eta_2 s} (1+s)^{-1/2} s^{-1/2} dy ds \le C,$$

by the integrability of $e^{-\eta_1|y|}$ and $e^{-\eta_2 s}$. Similarly, for the nonlinearity $(1+s)^{1/2}s^{-1/2}\psi_1^2$, we use the inequality

$$e^{-\eta|y|}(1+|y-a_j^- s|)^{-1} \le C\Big[e^{-\eta_1|y|}e^{-\eta_2 s} + e^{-\eta|y|}(1+s)^{-1}\Big],$$

from which the required estimate follows from the integrability of $e^{-\eta_1|y|}$ and of $(1+s)^{-2}$. For the nonlinearity $(1+s)^{1/2}s^{-1/2}\psi_2^2$, we have $y \in (-\infty, a_1^- s]$, for which exponential $|y|$ decay yields exponential s decay and the estimate is immediate. For the nonlinearity $(1+s)^{-1}(\theta+\psi_1+\psi_2)$, we observe the inequality

$$e^{-\eta|y|}(\theta+\psi_1+\psi_2)(y,s) \le C e^{-\eta_1|y|}(1+s)^{-1}.$$

We have, then an estimate by

$$C\int_0^\infty\int_{-\infty}^0 e^{-\eta_1|y|}(1+s)^{-2}dyds \le C_1\int_0^\infty (1+s)^{-2}\le C_2.$$

For the integral

$$\int_0^t\int_{-\infty}^0 |e_y(y,t-s)-e_y(y,+\infty)|\Psi(y,s)dyds,$$

we have from Remark 6

$$|e_y(y,t)-e_y(y,+\infty)|\le Ct^{-1/2}\sum_{a_k^->0} e^{-\frac{(y+a_k^-t)^2}{Mt}}.$$

Observing that our estimate on $|e_y(y,t)-e_y(y,+\infty)|$ takes the form of the Lax convection kernel multiplied by $(t-s)^{1/2}$, we proceed exactly as there to determine an estimate by $C(1+t)^{-1/2}$.

For the integral

$$\int_t^\infty |e_y(y,t-s)|\Psi(y,s)dyds,$$

we have from Remark 6

$$|e_y(y,t)|\le Ct^{-1/2}\sum_{a_k^->0} e^{-\frac{(y+a_k^-t)^2}{Mt}} + C\gamma e^{-\eta|y|}\Big(\mathrm{errfn}(\frac{y+a_k^-t}{\sqrt{4\beta_k^-t}})-\mathrm{errfn}(\frac{y-a_k^-t}{\sqrt{4\beta_k^-t}})\Big).$$

For the first, we can proceed simiarly as in the Lax convection estimates for $x=0$. For the second, we observe the inequality

$$e^{-\eta|y|}\Psi(y,s)\le Ce^{-\eta_1|y|}(1+s)^{-3/2}s^{-1/2}.$$

We estimate, then

$$\int_t^\infty\int_{-\infty}^0 e^{-\eta_1|y|}(1+s)^{-3/2}s^{-1/2}dyds \le C(1+t)^{-1/2}\int_t^\infty (1+s)^{-1}s^{-1/2}ds\le C'(1+t)^{-1/2}.$$

This completes the proof of Lemma 3

∎

Proof of Lemma 4. Observing that the first three estimates of Lemma 4 are significantly less precise than the second three, and moreover can be established by similar methods, we begin with the fourth,

$$\int_0^t\int_{-\infty}^0 |\tilde{G}_y(x,t-s;y)|\Phi_2(y,s)dyds,$$

where

$$\Phi_2(y,s)\le Ce^{-\eta|y|}(1+s)^{-3/2}.$$

Convection estimate. We consider integrals of the form

$$\int_0^t\int_{-\infty}^0 (t-s)^{-1/2}e^{-\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}}e^{-\eta|y|}(1+s)^{-3/2}dyds.$$

In order to make use of the localization due to the term $e^{-\eta|y|}$, we use the inequality

$$e^{-\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}}e^{-\eta|y|}\le C\Big[e^{-\frac{(x-a_k^-(t-s))^2}{M'(t-s)}}e^{-\eta|y|} + e^{-\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}}e^{-\eta_1|y|}e^{-\eta_2|x-a_k^-(t-s)|}\Big]. \tag{91}$$

For the first of these two estimates, we have the integral

$$\int_0^t \int_{-\infty}^0 (t-s)^{-1/2} e^{-\frac{(x-a_k^-(t-s))^2}{M'(t-s)}} e^{-\eta|y|}(1+s)^{-3/2}dyds.$$

In the case $|x| \geq |a_1^-|t$, we have

$$\begin{aligned} x - a_k^-(t-s) &= (x-a_1^-t) - (a_k^- - a_1^-)t + a_k^- s \\ &\leq (x-a_1^-t) \leq 0. \end{aligned}$$

We have, then, an estimate by

$$\begin{aligned} &C_1 t^{-1/2} e^{-\frac{(x-a_1^-t)^2}{M't}} \int_0^{t/2} (1+s)^{-3/2} ds \\ &\quad + C_2(1+t)^{-3/2} e^{-\frac{(x-a_1^-t)^2}{M't}} \int_{t/2}^t (t-s)^{-1/2} ds \\ &\quad \leq C(1+t)^{-1/2} e^{-\frac{(x-a_1^-t)^2}{M't}}. \end{aligned}$$

For $|x| \leq |a_1^-|t$, we write

$$x - a_k^-(t-s) = (x - a_k^- t) + a_k^- s.$$

In the event that $a_k^- > 0$, we have $|x - a_k^- t| \geq c(|x|+t)$, so that

$$(1+s)^{-3/2} e^{-\frac{(x-a_k^-(t-s))^2}{M'(t-s)}} \leq C(|x|+t)^{-3/2}.$$

In this case, we have the estimate

$$C(|x|+t)^{-3/2} \int_0^t (t-s)^{-1/2} ds \leq C(|x|+t)^{-1}.$$

For $a_k^- < 0$, we first consider the case $|x| \geq |a_k^-|t$, for which we again write

$$x - a_k^-(t-s) = (x - a_k^- t) + a_k^- s.$$

Observing that there is no cancellation between these terms, we can conclude an estimate by $C\theta(x,t)$ as in the case $|x| \geq |a_1^-|t$ above. In the event $|x| \leq |a_k^-|t$, we integrate (for $\epsilon > 0$ sufficiently small)

$$\begin{aligned} &\int_0^t \int_{-\infty}^0 (t-s)^{-1/2} e^{-\frac{(x-a_k^-(t-s))^2}{M'(t-s)}} e^{-\eta|y|}(1+s)^{-3/2}dyds \\ &\leq C_1 t^{-1/2} \int_0^{\epsilon|x-a_k^-t|} e^{-\frac{(x-a_k^-t)^2}{M't}} (1+s)^{-3/2} ds \\ &+ C_2 t^{-1/2} \int_{\epsilon|x-a_k^-t|}^{C|x-a_k^-t|\wedge t/2} (1+s)^{-3/2} ds \\ &+ C_3 t^{-1/2} \int_{C|x-a_k^-t|\wedge t/2}^{t/2} e^{-\frac{(x-a_k^-t)^2}{M't}} (1+s)^{-3/2} ds \\ &+ C_4 (1+t)^{-3/2} \int_{t/2}^t (t-s)^{-1/2} \\ &\leq C_1(1+t)^{-1/2} e^{-\frac{(x-a_k^-t)^2}{M't}} + C_2(1+t)^{-1/2}(1+|x-a_k^-t|)^{-1/2}. \end{aligned}$$

For integration over the second estimate in (91), we proceed similarly.

In this case the analyses of the reflection estimate and the transmission estimate do not differ significantly from the analysis of the convection estimate.

Overcompressive excited estimates. For the integral

$$\int_0^t \int_{-\infty}^0 |e_t(y,t-s)|e^{-\eta|y|}(1+s)^{-3/2}dyds,$$

we have an estimate by

$$C\int_0^t \int_{-\infty}^0 (t-s)^{-1/2}e^{-\frac{(y+a_k^-(t-s))^2}{M(t-s)}}e^{-\eta|y|}(1+s)^{-3/2}dyds,$$

for which we observe the inequality

$$e^{-\frac{(y+a_k^-(t-s))^2}{M(t-s)}}e^{-\eta|y|} \le C\Big[e^{-\eta_1|y|}e^{-\eta_2(t-s)}\Big].$$

We have, then

$$\begin{aligned}&\int_0^t \int_{-\infty}^0 (t-s)^{-1/2}e^{-\eta_1|y|}e^{-\eta_2(t-s)}(1+s)^{-3/2}dyds\\ &\le C_1e^{-\frac{\eta_2}{2}t}\int_0^{t/2}(1+s)^{-3/2}ds\\ &+C_2(1+t)^{-3/2}\int_{t/2}^t (t-s)^{-1/2}e^{-\eta_2(t-s)}ds\\ &\le C(1+t)^{-3/2}.\end{aligned}$$

For the final integral

$$\int_0^t \int_{-\infty}^0 |e(y,t-s)-e(y,+\infty)|e^{-\eta|y|}(1+s)^{-3/2}dyds,$$

we have an estimate by

$$C\int_0^t \int_{-\infty}^0 \mathrm{errfn}\Big(\frac{|y|-a(t-s)}{M\sqrt{t-s}}\Big)e^{-\eta|y|}(1+s)^{-3/2}dyds,$$

for some constants $M>0$, $a>0$. In this case, we observe the inequality

$$\mathrm{errfn}\Big(\frac{|y|-a(t-s)}{M\sqrt{t-s}}\Big)e^{-\eta|y|} \le Ce^{-\eta_1|y|}e^{-\eta_2(t-s)},$$

from which the estimate follows as above.

The remaining cases of Lemma 4 can be analyzed similarly. ∎

Acknowledgements. Thanks to Benjamin Texier for helpful discussions on the details of the fixed-point iteration scheme and to Dan Marchesin for pointing out the interest of multi-phase profiles with one endstate lying on the degenerate viscosity saturation boundary (Remark 4). Research of P.H. was supported in part by the National Science Foundation under Grant No. DMS–0230003. Research of K.Z. was supported in part by the National Science Foundation under Grants No. DMS-0070765 and DMS-0300487.

References

[1] A. Azevedo, D. Marchesin, B. Plohr and K. Zumbrun, Nonuniqueness of Riemann solutions, Z. Angew. Math. Phys. 47 (1996), no. 6, 977–998.

[2] A. Azevedo, D. Marchesin, B. Plohr and K. Zumbrun, *Bifurcation of nonclassical viscous shock profiles from the constant state*, Comm. Math. Phys. 202 (1999), no. 2, 267–290.

[3] A. Azevedo, D. Marchesin, B. Plohr and K. Zumbrun, *Capillary instability in multiphase flow models,* Z. Angew. Math. Phys. 53 (2002) 713–746.

[4] S. Benzoni–Gavage, D. Serre, and K. Zumbrun, *Alternate Evans functions and viscous shock waves*, SIAM J. Math. Anal. 32 (2001), 929–962.

[5] L. Q. Brin, *Numerical testing of the stability of viscous shock waves*, Ph.D. dissertation, Indiana University, May 1998.

[6] L. Q. Brin, *Numerical testing of the stability of viscous shock waves,* Math. Comp. 70 (2001) 235, 1071–1088.

[7] L. Brin, *Numerical testing of the stability of viscous shock waves*, Doctoral thesis, Indiana University (1998).

[8] L. Brin and K. Zumbrun, *Analytically varying eigenvectors and the stability of viscous shock waves*, to appear, Mat. Contemp. (2003).

[9] T. Bridges, G. Derks, and G. Gottwald, *Stability and instability of solitary waves of the fifth-order KdV equation: a numerical framework,* Phys. D 172 (2002), no. 1-4, 190–216.

[10] H. Freistühler, *Some results on the stability of non-classical shock waves,* J. Partial Diff. Eqs. 11 (1998), 23-38.

[11] H. Freistühler and P. Szmolyan, *Spectral stability of small shock waves,* Arch. Ration. Mech. Anal. 164 (2002) 287–309.

[12] H. Freistühler and K. Zumbrun, *Examples of unstable viscous shock waves,* unpublished note, Institut für Mathematik, RWTH Aachen, February 1998.

[13] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Englewood Cliffs, NY (1964), Reprint Ed. (1983).

[14] R. Gardner and K. Zumbrun, *The Gap Lemma and geometric criteria for instability of viscous shock profiles*, Comm. Pure Appl. Math. 51 (1998), no. 7, 797–855.

[15] M.G. Garroni and J.-L. Menaldi, *Green functions for second order parabolic integro-differential problems,* Pitman Research Notes in Mathematics Series, 275. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1992. xvi+417 pp. ISBN: 0-582-02156-1.

[16] J. Glimm, *Solutions in the large for nonlinear hyperbolic systems of equations,* Comm. Pure Appl. Math. 18 (1965) 697–715.

[17] J. Goodman, *Nonlinear asymptotic stability of viscous shock profiles for conservation laws,* Arch. Rational Mech. Anal. 95 (1986), no. 4, 325–344.

[18] J. Goodman, *Remarks on the stability of viscous shock waves*, in: Viscous profiles and numerical methods for shock waves (Raleigh, NC, 1990), 66–72, SIAM, Philadelphia, PA, (1991).

[19] J. Goodman and J. Miller, *Long-time behavior of scalar viscous shock fronts in two dimensions,* J. Dynam. Differential Equations 11 (1999), no. 2, 255–277.

[20] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, Springer–Verlag, Berlin (1981), iv + 348 pp.

[21] D. Hoff and K. Zumbrun, *Asymptotic behavior of multi-dimensional scalar viscous shock fronts,* Indiana Univ. Math. J. 49 (2000) 427–474.

[22] P. Howard and C. Hu, *Pointwise Green's function estimates toward stability for multidimensional fourth order viscous shock fronts,* Preprint 2004.

[23] P. Howard and K. Zumbrun, *Pointwise estimates for dispersive-diffusive shock waves,* Arch. Ration. Mech. Anal. 155 (2000) 85–169.

[24] J. Humpherys and K. Zumbrun, *Spectral stability of small amplitude shock profiles for dissipative symmetric hyperbolic–parabolic systems,* Z. Angew. Math. Phys. 53 (2002) 20–34.

[25] E.L. Isaacson, D. Marchesin and B.J. Plohr, *Transitional waves for conservation laws,* SIAM J. Math. Anal. 21 (1990), no. 4, 837–866.

[26] T. Kapitula, *Stability of weak shocks in λ–ω systems*, Indiana Univ. Math. J. 40 (1991), no. 4, 1193–12.

[27] T. Kapitula, *On the stability of travelling waves in weighted L^∞ spaces*, J. Diff. Eqs. 112 (1994), no. 1, 179–215.

[28] S. Kawashima and A. Matsumura, *Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion,* Comm. Math. Phys. 101 (1985), no. 1, 97–127.

[29] S. Kawashima, A. Matsumura, and K. Nishihara, *Asymptotic behavior of solutions for the equations of a viscous heat-conductive gas,* Proc. Japan Acad. Ser. A Math. Sci. 62 (1986), no. 7, 249–252.

[30] T.-P. Liu, *Pointwise convergence to shock waves for viscous conservation laws*, Comm. Pure Appl. Math. 50 (1997), no. 11, 1113–1182.

[31] T.P. Liu and K. Zumbrun, *Nonlinear stability of an undercompressive shock for complex Burgers equation,* Comm. Math. Phys. 168 (1995), no. 1, 163–186.

[32] T.P. Liu and K. Zumbrun, *On nonlinear stability of general undercompressive viscous shock waves,* Comm. Math. Phys. 174 (1995), no. 2, 319–345.

[33] A. Majda, *The stability of multi-dimensional shock fronts – a new problem for linear hyperbolic equations,* Mem. Amer. Math. Soc. 275 (1983).

[34] A. Majda, *The existence of multi-dimensional shock fronts,* Mem. Amer. Math. Soc. 281 (1983).

[35] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables,* Springer-Verlag, New York (1984), viii+ 159 pp.

[36] A. Matsumura and K. Nishihara, *On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas,* Japan J. Appl. Math. 2 (1985), no. 1, 17–25.

[37] C. Mascia and K. Zumbrun, *Pointwise Green's function bounds and stability of relaxation shocks*, Indiana Univ. Math. J. 51 (2002), no. 4, 773–904.

[38] C. Mascia and K. Zumbrun, *Stability of small-amplitude shock profiles of symmetric hyperbolic–parabolic systems,* Comm. Pure and Appl. Math., to appear.

[39] C. Mascia and K. Zumbrun, *Pointwise Green's function bounds for shock profiles with degenerate viscosity,* Arch. Rational Mech. Anal. 169 (2003), no. 3, 177–263.

[40] C. Mascia and K. Zumbrun, *Stability of large-amplitude shock profiles of hyperbolic–parabolic systems*, Arch. Rational Mech. Anal. 172 (2004), no. 1, 93–131.

[41] C. Mascia and K. Zumbrun, *Stability of large-amplitude shock profiles of general relaxation systems*, preprint (2003).

[42] R.L. Pego, *Phase transitions in one-dimensional nonlinear viscoelasticity: admissibility and stability,* Arch. Rational Mech. Anal. 97 (1987) 353–394.

[43] R. Plaza and K. Zumbrun, *An Evans function approach to spectral stability of small-amplitude viscous shock profiles,* preprint (2002).

[44] M.-R. Raoofi, *L^1-asymptotic behavior of perturbed viscous shock profiles,* Thesis, Indiana University (2004).

[45] D. Sattinger, *On the stability of waves of nonlinear parabolic systems*, Adv. Math. 22 (1976) 312–355.

[46] M. Slemrod, *The viscosity-capillarity approach to phase transitions,* in: *PDEs and continuum models of phase transitions* (Nice, 1988), 201–206, Lecture Notes in Phys., 344, Springer, Berlin-New York, 1989.

[47] M. Slemrod, *The vanishing viscosity-capillarity approach to the Riemann problem for a van der Waals fluid,* in: *Nonclassical continuum mechanics* (Durham, 1986), 325–335, London Math. Soc. Lecture Note Ser., 122, Cambridge Univ. Press, Cambridge-New York, 1987.

[48] M. Slemrod, *A limiting "viscosity" approach to the Riemann problem for materials exhibiting change of phase,* Arch. Rational Mech. Anal. 105 (1989), no. 4, 327–365.

[49] M. Slemrod, *Dynamic phase transitions in a van der Waals fluid,* J. Differential Equations 52 (1984), no. 1, 1–23.

[50] M. Slemrod, *Admissibility criteria for propagating phase boundaries in a van der Waals fluid,* Arch. Rational Mech. Anal. 81 (1983), no. 4, 301–315.

[51] A. Szepessy and Z. Xin, *Nonlinear stability of viscous shock waves,* Arch. Rat. Mech. Anal. 122 (1993) 53–103.

[52] A. Szepessy and K. Zumbrun, *Stability of rarefaction waves in viscous media,* Arch. Rational Mech. Anal. 133 (1996), no. 3, 249–298.

[53] K. Zumbrun and P. Howard, *Pointwise semigroup methods and stability of viscous shock waves,* Indiana Mathematics Journal V47 (1998) no. 4, 741–871.

[54] K. Zumbrun, *Stability of viscous shock waves*, Lecture Notes, Indiana University (1998).

[55] K. Zumbrun, *Refined Wave–tracking and Nonlinear Stability of Viscous Lax Shocks*, Methods Appl. Anal. 7 (2000) 747–768.

[56] K. Zumbrun, *Multidimensional stability of planar viscous shock waves*, Advances in the theory of shock waves, 307–516, Progr. Nonlinear Differential Equations Appl., 47, Birkhuser Boston, Boston, MA, 2001.

[57] K. Zumbrun, *Stability of large-amplitude shock waves for compressible Navier–Stokes equations,* preprint (2003).

[58] K. Zumbrun, *Planar stability criteria for viscous shock waves of systems with real viscosity,* CIME summer school notes, preprint (2004).

[59] K. Zumbrun, *Dynamical stability of phase transitions in the p-system with viscosity-capillarity,* SIAM J. Appl. Math. 60 (2000) 1913–1924.

[60] K. Zumbrun and D. Serre, *Viscous and inviscid stability of multidimensional planar shock fronts,* Indiana Univ. Math. J. 48 (1999) 937–992.

Peter HOWARD
Department of Mathematics
Texas A&M University
College Station, TX 77843
phoward@math.tamu.edu
and
Kevin ZUMBRUN
Department of Mathematics
Indiana University

Bloomington, IN 47405-4301
kzumbrun@indiana.edu